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Manipulatives and the Growth of Mathematical Understanding

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A thesis submitted to the faculty of  
Brigham Young University  
In partial fulfillment of the requirements for the degree of

Master of Arts

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## ABSTRACT

### Manipulatives and the Growth of Mathematical Understanding

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The purpose of this study was to describe how manipulatives facilitated the growth of one group of high school students' mathematical understanding of combinatorics and Pascal's Triangle. The role of manipulatives in mathematics education has been extensively studied, but much of the interest in manipulatives is focused on the general uses of manipulatives to support student learning. Unfortunately, there is a lack of research that explicitly defines how manipulatives can help students develop mathematical understanding. I have chosen to examine mathematical understanding through the lens of the Pirie-Kieren Theory for Growth of Mathematical Understanding. Through analysis of the students' explorations of the Towers Task, I identified ways in which manipulatives facilitated students' understanding of combinatorics and Pascal's Triangle. It was found that the properties and arrangements of the manipulatives were significant in prompting students' progression through levels of understanding and helped students to reason abstractly and develop mathematical generalizations and theories. From this study we can gain insights into explicit ways in which manipulatives facilitate mathematical understanding. These results have implications for research, teaching and teacher education.

Keywords: [manipulatives, understanding, Pirie-Kieren, Pascal's Triangle]

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## Chapter 1: Introduction

The ways in which mathematical ideas are represented are fundamental to how students can understand those mathematical ideas. The National Council of Teachers of Mathematics (2000) advocates the development of representations of particular mathematics concepts in order to deepen students' understanding of those concepts. The representations students produce and use are essential in supporting students' understanding of mathematical concepts (diSessa & Sherin, 2000; Richardson, Berenson, & Staley, 2009; Speiser, Walter, & Maher, 2003), in helping students develop and convey ideas (Morris & Speiser, 2010; Speiser, Walter, & Sullivan, 2007), and in recognizing connections between related concepts (Brenner, Herman, Ho, & Zimmer, 1999; Warner, 2008). As students develop a variety of representations of mathematical ideas, they have a set of tools they can use to help expand their ability to think mathematically.

Students of all ages can create representations, but high school students in particular should develop a larger set of mathematical representations and knowledge of how to use those representations as they become more mathematically sophisticated. High school students should be able to create and interpret representations of more complex mathematical ideas and also be able to convert flexibly among various types of representations such as graphs, tables of data, algebraic expressions and physical representations (NCTM, 2000). Although the NCTM (2000) advocates the ability of high school students to create and move flexibly among representations of mathematical ideas, there remains a need to show how high school students learn to move flexibly among specific types of representations and how high school students build understanding from those representations.

Most research on students' development and use of representations deals with elementary, middle school and college-aged students, but some research has addressed high

school students' creation of representations. Researchers have studied high school students' creation and use of graphs (Parzysz, 1991), tables of data (Speiser, et al., 2003) algebraic expressions (Moschkovich, Schoenfeld, & Arcavi, 1993) and drawings or pictures (Azevedo, 2000; Sherin, 2000). However, very little has been said about high school students' use of manipulatives, a particular type of physical representation.

The role of manipulatives in mathematics education has been studied extensively throughout the past twenty years, and researchers agree that manipulatives can help students to make sense of abstract mathematical concepts (Chao, Stigler, & Woodward, 2000; Chappell & Strutchens, 2001; Fuson & Briars, 1990). Much of the interest in manipulatives stems from the assumption that their concrete nature makes them particularly appropriate for young students as they use manipulatives to develop mathematical meaning for concepts (Manches, O'Malley, & Benford, 2010; McNeil & Jarvin, 2007). However, there is a lack of research on the effectiveness of manipulatives in high school settings where students are likely to use manipulatives differently than elementary students. Additionally, much of the research on manipulatives focuses on ways in which manipulatives are useful, while the ways in which they help students develop understanding has not been addressed. There is a need to show whether manipulatives can help high school students to build mathematical understanding and how they may do so.

This research centers on high school Algebra 2 students as they engage in collaborative problem solving of a combinatorics task. These students were part of a teaching experiment in which pre-service teachers team-taught their Algebra 2 class as part of the pre service teachers' practicum course. The research presented here focuses on one group of three students, Beth, Ashley and Jamie, as they grapple with the Towers Task. The Towers Task is designed to elicit ideas about combinations and Pascal's Triangle by prompting students to build block towers of

particular heights when given two colors of blocks to choose from. I analyzed how these students used manipulatives to build understanding over time, in order to determine how manipulatives are useful in building understanding.

## **Chapter 2: Theoretical Perspective and Literature Review**

Since I am interested in addressing how manipulatives facilitate students' growth of mathematical understanding, I have based my study on the guiding principle that learners must adapt and build their understanding in order to explore and address new problems where their current understanding proves inadequate. I view the notion of building mathematical understanding as a dynamic process, which requires students to move back and forth between levels of sophistication in their thinking. The representations students create can help them build mathematical understanding and communicate their understanding to others. In particular, manipulatives are a unique representation that may support students' development of mathematical understanding of ideas and concepts.

This chapter is divided into three main sections. In the first section I will discuss mathematical understanding, specifically the Pirie-Kieren Theory for growth of mathematical understanding. In the second section, I will provide definitions of mathematical representations and manipulatives as well as a summary of prior research pertaining to these topics. In the third section, I will discuss mathematical concepts that can be elicited by the Towers Task and provide a summary of other research on the Towers Task. Finally, I will summarize how these three main categories are related to the research problem of this study and introduce my research question.

### **Section 1: Mathematical Understanding**

In this section, I will outline several prominent theories of understanding that have been developed and explain why I have chosen to use the Pirie-Kieren Theory for understanding. I will describe the Pirie-Kieren Theory in detail and then provide a summary of some of the past research that has been done pertaining to the theory.

**Mathematical Understanding.** Although the notion of “understanding” gets used freely throughout mathematics education research, the goal to develop a concise definition of understanding has continued for many years. This has resulted in several different frameworks for types of understanding. The difficulty of attempting to define understanding in the mathematics education community arose after Skemp (1978) introduced instrumental and relational understanding. Prior to Skemp’s influential paper, researchers generally identified understanding as the ability to perform algorithmic operations (Meel, 2003). Skemp’s (1978) work made a distinction between knowledge and understanding and elaborated on two categories of understanding: relational and instrumental. Skemp described relational understanding as “knowing both what to do and why” (p. 9) and instrumental understanding as “rules without reasons” (p. 9). Instrumental understanding tends to promote immediate rewards and to provide quick access to answers. On the other hand, relational understanding provides a foundation for more efficient transfer, for facilitating the growth of understanding.

Byers and Herscovics (1977) built upon Skemp’s work and introduced four different types of understanding: instrumental, relational, intuitive and formal. Like Skemp, they defined instrumental understanding as the ability to apply a memorized rule to solve a problem without knowing why the rule works and relational understanding as the ability to determine specific rules or procedures from more general mathematical relationships. They define intuitive understanding as the ability to solve a problem without prior experience and formal understanding as the ability to connect mathematical notation with relevant mathematical ideas and to combine the ideas into chains of logical reasoning.

Buxton (1977) also suggested four different levels, or types of understanding: rote, observational, insightful and formal. Buxton defined the first level “rote” as being purely



instrumental, where students memorize procedures, which are reinforced by exercises that require immediate recall of facts. The observational level is not purely instrumental, but it is not fully relational. In this level, students are able to see patterns in procedures, which can help them to make more generalized statements. Buxton defined the third level, “insightful” as being equivalent to relational understanding where students know how to use a procedure and why the procedure is effective. Finally, in the formal level, students can recognize the need for proof and justification in mathematics, which is usually only achieved after developing insightful understanding.

Another prominent categorization of understanding can be found in two strands of the National Research Council’s (2001) strands of mathematical competency: conceptual understanding and procedural fluency. Conceptual understanding is the ability to recognize why a mathematical idea is important and the kinds of contexts in which it would be useful. Students with conceptual understanding have organized their knowledge into a coherent whole and are able to connect new ideas with ideas they already have knowledge of. Procedural fluency is the knowledge of how to use procedures appropriately and the skill in performing procedures flexibly, accurately and efficiently.

These frameworks, and others, have tried to clarify the idea of understanding by categorizing understanding into different types or levels. Although these types and levels of understanding can be useful in characterizing the complexities and components of understanding, they still leave questions to be answered about the process of understanding. Sierpiska (1990) raised the following questions about understanding. “Are there levels, degrees or rather kinds of understanding? ... Is understanding an act, an emotional experience, an intellectual process, or a way of knowing? ... What are the conditions for understanding as an act to occur? ... How do

we come to understand? ... Can understanding be measured and how?" (p. 24). Pirie and Kieren (1994) consider their work on the growth of mathematical understanding to provide answers to some of the questions raised by Sierpinska.

Pirie and Kieren's (1994) Theory for growth of mathematical understanding will provide the underlying structure for examining students' processes of building understanding. Pirie and Kieren's (S. B. Pirie, 1989) initial definition of mathematical understanding evolved from von Glasersfeld's constructivist view of understanding. In particular, von Glasersfeld (1987) introduced the following definition of understanding:

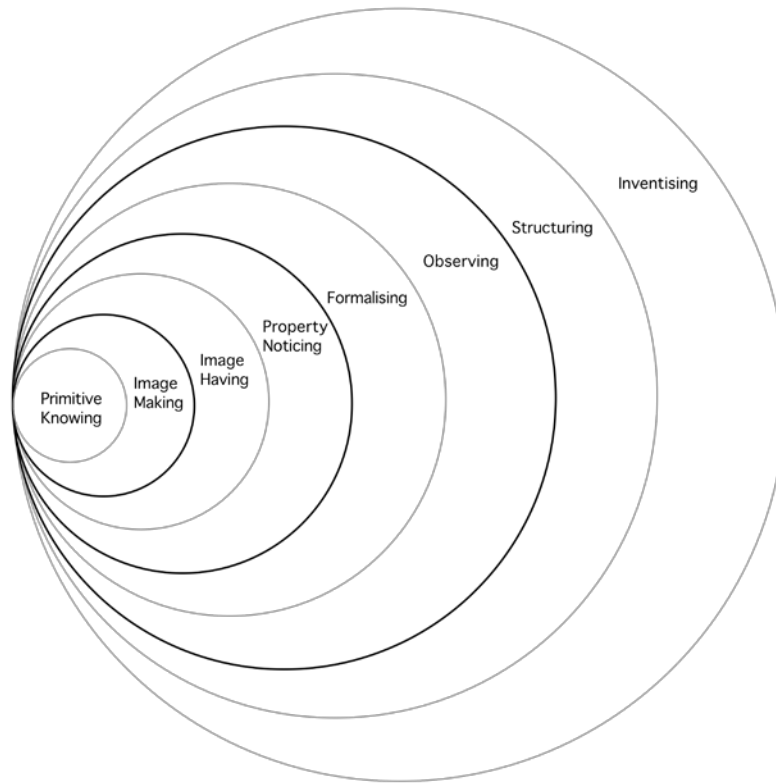
The experiencing organism now turns into a builder of cognitive structures, intended to solve such problems as the organism perceives or conceives ... among which is the never ending problem of consistent organizations [of cognitive structures] that we call understanding (p. 7)

von Glasersfeld viewed understanding as the continual problem solving process of organizing one's cognitive structures. Using this definition as a foundation, Pirie and Kieren (1991) began to develop their theoretical view concerning mathematical understanding. Pirie and Kieren (1991), view mathematical understanding as a complex phenomenon that cannot be easily differentiated into two or three categories or identified as a single acquisition. They describe mathematical understanding as a leveled but non-linear, recursive phenomenon as one's thinking moves between levels of sophistication. Each level of understanding is contained within succeeding levels and depends on previous levels (S. Pirie & Kieren, 1989). Their theory elaborates the nature of understanding as the personal building and re-organization of an individual or group's knowledge structures.

I have chosen to use the Pirie-Kieren Theory for growth of understanding for a couple of reasons. Like Pirie and Kieren, I believe that mathematical understanding is a complex process that cannot be categorized into one or two "types" of understanding such as relational and

instrumental. I share the same view that mathematical understanding is not a static acquisition that can be achieved. I am drawn to the way that Pirie and Kieren view mathematical understanding as a continual movement through many levels of sophistication and that they take into account the entire process of building understanding rather than the finished product.

**Pirie-Kieren Model for Growth of Mathematical Understanding.** Central to the Pirie-Kieren's Theory for growth of understanding is a model (Fig. 1) they use to demonstrate eight potential levels of understanding that students may traverse as they build mathematical understanding (S. Pirie & Kieren, 1989; S. Pirie & T. Kieren, 1994; S. B. Pirie & Kieren, 1992). These levels are *primitive knowing*, *image making*, *image having*, *property noticing*, *formalizing*, *observing*, *structuring* and *inventising*. The model is presented as a sequence of eight embedded circles, where each circle represents a level of understanding. The model emphasizes that each level of understanding contains all previous levels, and is embedded in all succeeding levels of understanding. Although the levels of understanding in the model grow outward, growth of understanding occurs with continual movement back and forth through the levels of understanding (L. C. Martin, 2008), and with an overall trend towards the outer levels of understanding. In this model, the term "growth" does not refer to a linear process, as in how a tree or plant grows, but instead growth of understanding refers to moving back and forth between levels of understanding, where any movement, even a backwards one, represents the development of understanding.



*Figure 1: The Pirie-Kieren model for the growth of mathematical understanding*

The first level, *primitive knowing*, does not imply low-level mathematics, but is rather the starting place for the growth of mathematical understanding (S. Pirie & T. Kieren, 1994). This level consists of all the knowledge that a student has when starting a new mathematical task or learning about a new concept. *Primitive knowing* is the foundation from which all other understanding is built, and is embedded in all other levels of understanding (Thom & Pirie, 2006).

In the second level, *image making*, the learner makes distinctions in their primitive knowledge and use it in new ways. Students are able to form images of their previous knowledge by performing some type of action. For example, students can begin to make images of a parabola by creating a table of values and plotting ordered pairs in order to graph the quadratic. The image making can be seen in comments like, “Is this the way that  $x^2 + 4x + 2$  is supposed

to look?” or “I get seven when I plug one in for x.” (S. B. Pirie, 1989). What is critical in this level is that actions of the student involve doing something, either mentally or physically, in order to develop initial understanding of a concept (Meel, 2003).

In the third level, *image having*, students use an image about a topic without necessarily having to do the particular action that brought about the image. This frees the student from having to perform particular actions associated with each image (S. Pirie & T. Kieren, 1994). This is a first form of abstraction, and the student does so by building on the images they made in *image making*. For example, students in *image having* would be able to talk generally about quadratics and note that they are all u-shaped curves without having to plot points to make the graph (S. B. Pirie, 1989). It is in the *image having* level where students begin to recognize general properties of the mathematical images they have created.

In *property noticing*, students begin to manipulate or combine aspects of the images they have constructed in previous levels in order to construct context specific, relevant properties about a given topic. This involves noting distinctions, combinations or connections between images and determining the relationships between these various details (S. B. Pirie & Kieren, 1992). The difference between *image having* and *property noticing* is the ability to notice a connection between images and explain how to verify the connection (Meel, 2003). Students in *property noticing* may realize that quadratic equations have vertices, that quadratics either open up or down depending on the value of the leading coefficient, or that they are not linear.

In the fifth level, *formalizing*, students are able to abstract a common quality from a previous image in order to develop formal mathematical ideas. It entails consciously thinking about the noted properties and abstracting the commonalities in those properties (S. Pirie & T. Kieren, 1994). In this level, students make generalizations about specific images and begin to

think in terms of general cases (Thom & Pirie, 2006). The noticed properties and generalizations of concepts made in this level can help students develop mathematical definitions. The language used to describe the definition does not need to be formal mathematical language, but the general descriptions provided by students should be equivalent to the appropriate mathematical definition (Meel, 2003). For example, students in the *formalizing* level realize that there is a set of quadratic functions of the form  $ax^2 + bx + c$ .

In the *observing* level, students begin to reflect on the formal activity in which they participated in the *formalizing* level and try to look for patterns and connections in order to define their formal ideas as algorithms or theorems (S. B. Pirie & T. E. Kieren, 1994). In this level, students are in a position to organize their ideas in a consistent and logical way and create theories based on their ideas. For example, after developing formal mathematical ideas about quadratics in the previous levels, students in the *observing* level may begin to see patterns and connections about their ideas and ask themselves what may be true for all quadratics such as, do all quadratics have real roots? (S. B. Pirie, 1989).

After developing their ideas into theories within the *observing* level, the natural expectation is to determine whether or not their theories are true. In the *structuring* level, students justify or verify their mathematical theories through justification and proof (S. Pirie & T. Kieren, 1994). For a fuller understanding, students must be able to answer *why* their formal observations may be true or *why* their observations may not be true. Students in the *structuring* level may be led to prove whether or not quadratics have real roots (S. B. Pirie, 1989).

In the final level, *inventising*, students are able to break away from the structured understanding that they have built in order to pose new questions that may lead to the process of building understanding of a different concept. At this level of understanding, a student's

mathematical understanding is unbounded and reaches beyond the current structure to contemplate the question of “what if?” (Meel, 2003). For example, students in the *inventising* level may raise the question “What is the relationship between  $y = ax^2 + bx + c$  and  $x = ay^2 + by + c$ , how are these equations similar or different?”

An important feature of the Pirie-Kieren theory of understanding is the notion of *folding back*. When a student encounters a problem, whose solution he does not know how to attain, the student must fold back to an inner level of understanding to extend the current, inadequate understanding (S. Pirie & T. Kieren, 1994). By reflecting and reorganizing his earlier construct for the concept, the student can create new images that will help him continue to build his mathematical understanding. Folding back does not imply students’ inability to progress, but rather provides students the opportunity to extend their understanding (L. C. Martin, 2008; Meel, 2003). The extension of a student’s understanding is not just a product of generalizing or abstracting the knowledge gained from the inner level in order to build understanding in an outer level. Instead, the progression of a student’s understanding occurs as they fold back to recursively reconstruct and reorganize their understanding gained in an inner level in order to extend to outer levels (Meel, 2003). The continual process of folding back to move forward throughout the model accounts for the dynamic feature of the Pirie-Kieren Theory.

**Research on Pirie Kieren Theory.** The Pirie-Kieren Theory has been used and extended in numerous studies concerning students’ growth of mathematical understanding. Several studies elaborate the effects that folding back has on students’ growth of understanding (L. Martin & Pirie, 1998; L. C. Martin, 2008). The Theory is also extended in different mathematical subjects such as operations with fractions (S. B. Pirie & Kieren, 1992) and combinatorics (Warner, 2008).

***Folding Back.*** As an important feature of the Pirie-Kieren Theory, the activity of folding back is the focus of several research studies. Folding back occurs when students are faced with a problem that is not immediately solvable with the level of understanding they are currently in. Martin and Pirie (1998) described how the activity of folding back can be classified as either effective or ineffective. Effective folding back occurs when a student folds back to an inner level of understanding and is able to use the resulting growth in understanding to solve the original problem. On the other hand, folding back is ineffective if the student folds back, but is unable to use the knowledge gained in the inner levels of understanding to solve the original problem. The authors claim that effective folding back is more likely when a student is given time to work with their inner levels of understanding in order to enrich the understanding before returning to the original problem they encountered in an outer level. Effective folding back requires students to re-evaluate and reorganize their current understanding instead of just recalling thoughts or ideas.

Martin (2008) further explained that folding back may be prompted by a teacher, a peer, or learners themselves. A teacher can prompt students to fold back by asking hypothetical questions such as “Are you sure...?” or “What if...”, teachers can also explicitly focus students on previous understanding or images that may be useful to students when solving problems. A peer can be the source of folding back by asking hypothetical questions or by suggesting that a student may need to rethink or adjust their ideas. Students themselves may be the source of folding back by recognizing they need to extend their understanding of a concept, which will help them to move forward in solving the problem.

Martin (2008) also provided examples of students folding back prompted by each source. A teacher prompted two students, who were having trouble finding the area of a triangle in order to approximate the area of a sector of a circle, to fold back to their basic understanding of the



area of triangles by explicitly asking the students how they might find the area of a triangle. When exploring what happens to various shapes in taxi-cab geometry, a peer prompted a student, Clare, to fold back to her primitive knowledge of taxi cab geometry by asking “How did your circle become a diamond?” and then allowing Clare to explain her thinking. When a student Rachel was trying to find the area of a sector of a circle, she did not have any immediate strategies to do so, but proceeded to look through her textbook to find the section about area. Rachel recognized she needed to extend her existing understanding of area and attempted to do so by finding the area section of her textbook.

**Mathematical Subjects.** The Pirie and Kieren Theory has been used to analyze students’ growth of understanding of particular mathematical subjects such as fractions and combinatorics. Pirie and Kieren (1992) used their model to analyze the growth of understanding of an eight year old boy, Sandy, who attempted to add fractions with non-equal denominators. As his understanding developed, Sandy moved from the *image making* level to the *formalizing* level of understanding. Sandy demonstrated the notion that fractions are numbers (*image having*) that can be added together (*property noticing*). In trying to determine how to add fractions together, Sandy developed a theory for adding together unit fractions. He stated that one could multiply the denominators to get the new denominator and add the denominators to get the new numerator (*formalizing*). The interviewer posed another problem,  $\frac{2}{3} + \frac{1}{5}$ , which caused Sandy to fold back to *image having* in order to make sense of what the fractions mean in this new problem. Sandy eventually developed his own general rule for adding any two fractions that he could apply without necessarily having to think about the meanings of the written symbols of the fractions (*formalizing*).

Warner (2008) used the Pirie-Kieren Theory in order to analyze how the problem solving behaviors of 6<sup>th</sup> grade students facilitated the growth of their understanding of a combinatorics task that was very similar to the Towers Task. The students were asked to find a number of combinations of striped flags they could make given two different color stripes to choose from. The students drew various representations of the different flags that helped them to create images of the problem (*image having*) and to notice properties (*property noticing*) and patterns (*observing*) about the problem. Students moved back and forth between levels of understanding as they worked from less sophisticated representations to more general and abstract representations of the problem. Warner mapped the students' movement across the different levels of understanding using the Pirie-Kieren model. She concluded that certain types of problem solving behaviors such as questioning, explaining ideas, linking representations and setting up hypothetical situations were associated with the growth of understanding.

## **Section 2: Representations and Manipulatives**

In the following section, I will provide definitions of mathematical representations and manipulatives. I will also discuss previous research done pertaining to mathematical representations and manipulatives. Finally, I will end the section by summarizing the research and suggesting how more general research findings might apply to students' development of mathematical understanding using manipulatives.

**Definition of Mathematical Representations.** There are many interpretations of the term “representation” in connection with mathematics teaching and learning. Goldin and Janvier (1998) summarized the discussion of the notion of mathematical representations that took place in the early 1990's. They described a mathematical representation as an external, internal or linguistic configuration or system that embodies mathematical ideas. Others have argued that

mathematical representations are tools or strategies which students use in order to process, express and understand mathematical situations and mathematical concepts (Barnby, Harries, Higgins, & Suggate, 2009; Brenner, 1995; Cifarelli, 1998; Ozgun-Koca, 1998; Richardson, et al., 2009). Some authors view representations as presentations that students use to describe specific details of their mathematical thinking (Morris & Speiser, 2010; Speiser, et al., 2003; Speiser, et al., 2007; Warner, 2008). These presentations could include pictures, tables, graphs or symbols that students use to communicate the meaning they have developed of particular mathematical topics to their teacher or peers.

I will combine these notions of mathematical representations to define my view of a mathematical representation as follows: a mathematical representation is tool that students use to assist them in creating meaning of mathematical concepts and to communicate that meaning to their peers or teacher. I do not believe that mathematical meaning is embedded in the representation, but rather that students develop meaning through creating, analyzing, explaining or using a representation in order to justify their mathematical ideas and conjectures.

**Research on Representations.** In reviewing the literature on mathematical representations, I found three main ways in which representations may help students build mathematical understanding. It is important to note that the authors of these studies discuss how representations can be used to develop understanding, but many of the authors do not explicitly define what they mean by understanding. I will summarize the literature on representations without addressing the lack of rigor in defining understanding in these studies or how their views of understanding necessarily relate to the Pirie-Kieren Theory. Later in my analysis, I will use the Pirie-Kieren Theory for understanding to elaborate on what understanding looks like as it is facilitated by students' use of representations, specifically manipulatives. I will then address how

attuning to both the Pirie-Kieren Theory and the use of representations can help make the connections between representations and mathematical understanding more explicit.

One way students can build understanding from the representations they develop is by making connections and moving flexibly between various representations (Barmby, et al., 2009; Speiser, et al., 2007). Speiser et al. (2003) provided an example of how several representations became an important means for developing pre-calculus students' ideas and conceptions of motion. The students constructed numerical tables and graphs based upon data from pictures of a moving object. As the students interpreted the tables and graphs, they connected the new information gleaned from the representations with their own physical experience of running. The connections the students made between the representations they developed from the pictures played a key role in the development of their understanding of motion.

Warner (2008) also demonstrated that flexibly using different representations on the same problem and making connections between them is evidence of mathematical understanding. The 6<sup>th</sup> grade students in the study used manipulatives, pictures and charts in order to solve several combinatorics problems. The students worked together to revise and make connections across the representations they had developed. By moving across these different representations, the students built a better understanding of the mathematical concepts the problems addressed. Warner also noted that as students moved across representations more consistently, they built deeper understanding.

A second way in which students can build understanding from representations is by using the representations in order to make justifications and generalizations about mathematical concepts. Representations can provide students with a means to question others' ideas and also justify and clarify their own ideas (Glass, 2004; Speiser, et al., 2007). Richardson et al. (2009)

showed how pre-service elementary teachers constructed representations of pattern-finding tasks, and used their representations in order to generalize the patterns using algebraic notation, and to justify the accuracy of the algebraic notation they developed. Morris and Speiser (2010) described how university calculus students created various representations that helped them estimate the slope in a particular problem. The students clarified and justified their representations, which in turn helped them to refine their representations and gain a deeper understanding of the problem. Students built their mathematical understanding by creating representations, and then using the representations in order to justify and generalize their mathematical ideas.

A third way in which students can develop understanding from representations is by using representations to mathematically model experiences from their own lives. Students can draw on their own experiences in order to mathematically represent various aspects of their lives, which can help them build mathematical understanding (Cifarelli, 1998; Sherin, 2000). Sherin (2000) demonstrated how middle school students drew on their own experiences with motion in order to mathematically represent the motion. Sherin argued that students possess a great deal of knowledge about representations from their experiences, and that they should be given opportunities to mathematically represent problems in order to build their understanding from their previous knowledge. Cifarelli (1998) noted that as students develop representations based on their experiences, they actively construct new knowledge about the problem situation as they reorganize and conceptualize their representations. Cifarelli studied first year college students who used their own experiences of space and area in order to create representations that they used to solve algebra story problems. The students built mathematical understanding based upon the representations they created from their personal experiences.

In summary, I found three main ways in which representations can help build students' mathematical understanding: 1) by making connections and moving flexibly between various representations, 2) by using the representations in order to make justifications and generalizations about mathematical concepts, and 3) by using representations to mathematically model experiences from their own lives. Although some literature has addressed how representations can help develop students' mathematical understanding, many of the authors did not explicitly define their view of understanding. I plan to use the Pirie-Kieren Theory in order to determine how a physical representation, manipulatives, can facilitate one group of high school students' growth of understanding of combinatorics concepts over time.

**Definition of Manipulatives.** Manipulatives are one form of representation that offers unique opportunities for students to build and communicate mathematical ideas. Ball (1992) defined manipulatives as concrete objects which students can use as a tool for exploring, representing and communicating mathematical ideas. Other definitions suggest that manipulatives are concrete materials which represent mathematical concepts and can be touched and moved around by students through hands on experience (Moyer, 2001; Namukasa, Stanley, & Tuchtie, 2009). Uttal, Scudder and DeLoach (1997) provided a different perspective on manipulatives, as they view manipulatives as symbols or representations of abstract mathematical concepts and the relationship between mathematical concepts. For the purposes of this study, I define manipulatives as a physical representation that can be manipulated by students to help justify and clarify their mathematical ideas.

**Research on Manipulatives.** Researchers have been endorsing the use of manipulatives for decades, dating back to Piaget (1952), Dienes (1960) and Bruner (1966), and much of this research has focused on the benefit of manipulatives for young children. The foundation for

theoretical discussions about manipulatives is frequently taken from the work of Piaget. Piaget's learning theory, which described the development from concrete to abstract reasoning, provided the theoretical foundations for the role of manipulatives in supporting students' learning.

Although Piaget's work was more a theory of the development of knowledge, rather than a theory of instruction, it did highlight the role of concrete materials in helping younger children develop understanding (Manches, et al., 2010). Piaget (1952) suggested that children do not have the mental maturity to grasp abstract mathematical concepts presented in words or symbols alone and may need experiences with concrete materials and drawings in order for learning to occur.

Piaget's theory of manipulatives found support in the work of Dienes and Bruner. Dienes (1960) asserted that learning is an active process and that mathematical ideas must be constructed. He believed the structure of a mathematical idea cannot be abstracted from concrete objects, but instead must be abstracted from the operational or organizational systems that one imposes on the concrete objects. It is only after the organization systems have been constructed that children can use manipulatives as a model that embodies the underlying structure. Dienes (1960) also identified the principle of multiple embodiment wherein mathematical ideas cannot be abstracted from single isolated patterns, but instead, abstraction occurs when students recognize structural similarities shared by several related models. Dienes believed that concrete materials offer multiple embodiments of mathematical concepts. For example, base-ten blocks are not enough when attempting to teach arithmetic regrouping operations. Students should also investigate other models such as bundling sticks or an abacus.

Bruner (1966) believed that elementary school children's thinking focused on concrete properties that could be actively manipulated. He suggested that children should progress from working with hands-on physical materials to iconic and ultimately symbolic representations.

Bruner argued that this approach would help students to move beyond the specific properties of manipulatives and help students to grasp the abstract properties the manipulative represented. Bruner specifically called for the use of concrete objects during instruction, suggesting that using many different concrete objects could help children focus on the perceptual properties of the individual objects.

Since the emergence of these theories advocating concrete materials, the use and role of manipulatives in the teaching and learning of mathematics has been the focus of many research studies. Several studies support the notion that manipulatives can help students to proceed from concrete representations to more formal and abstract representations (Chao, et al., 2000; Chappell & Strutchens, 2001; Fuson & Briars, 1990). Other studies claim that manipulatives can be an important tool in helping students to relate their own physical and real life experiences to mathematics (Carraher, Carraher, & Schliemann, 1985; Kamii, Lewis, & Kirkland, 2001). Some research supports the use of manipulatives as they can help engage students in mathematical activity (Moch, 2001; Moyer, 2001). Finally, others argue that the manipulability and concrete nature of manipulatives may promote student learning (Clements, 2000; Hinzman, 1997; Resnick, 1998).

**Promote Abstract Reasoning.** The idea that manipulatives can help students to move from concrete reasoning to more abstract reasoning has been extensively studied. Chao et al. (2000) found that students who used manipulatives with a 10-frame structure (such as a pack of candy that held 10 candies) demonstrated more sophisticated strategies in solving multidigit addition and subtraction problems and began to develop base ten understanding of quantities. Fuson & Briars (1990) also studied the effect that base ten blocks had on young children. They found that students who solved multidigit addition and subtraction problems using manipulatives



developed understanding of place value and verbally explained the carrying and borrowing operations by demonstrating the procedures with blocks. Chappell and Strutchens (2001) showed how eighth-grade algebra students used concrete manipulatives when learning about polynomials. The students used algebra tiles to represent the product of two binomials, and made connections between the concrete models and the algebraic steps when completing the multiplication with paper and pencil.

Some research has suggested that although manipulatives can promote abstract reasoning, students should not have to rely on the manipulatives once they have started to reason abstractly. According to Piaget (1952), cognitive development starts with the use of physical actions or concrete models, followed by the use of symbols after that. Therefore, students would begin to develop understanding by using manipulatives, but then replace the manipulatives with symbolic representations as their understanding becomes more advanced. Bruner (1966) argued that the sequence of knowledge should progress from using concrete objects to form mathematical conceptions, which would then lead to using abstract symbols. Ambrose (2002) explained that children often use manipulatives to initially solve single-digit arithmetic, but then progress from using concrete strategies to abstract strategies that do not involve using manipulatives. Ambrose went on to point out that not all students progress from concrete to abstract strategies, and manipulatives may hinder some students' abilities to develop more sophisticated abstract strategies to solve problems.

***Real-Life Experiences.*** Manipulatives can help children draw on their practical, real-world knowledge. An example of this is Carraher, Carraher, and Schliemann (1985), who showed that Brazilian children, who sold goods at street markets, performed well on math problems presented in the familiar buying-and-selling street context of physical materials such as

coconuts or lemons, but could not solve the same types of problems presented in traditional mathematics classroom settings. The students' experience of buying and selling goods, the manipulatives, helped them to generalize the arithmetic in that context and abstract what they learned from the manipulatives to do mental math. The physical context of buying and selling goods was a reference point for the students and helped them to make sense of the mathematics that related to this real-world context. Kamii, Lewis and Kirkland (2001) pointed out that children can learn  $3 + 5 = 8$  by hanging weights of "3" and "5" on one side of a balance and "8" on the other side. The balance uses weight, which is real-world knowledge, in order to help students to understand aspects of equations and equality. These authors also provided the example of students using tangrams in order to develop spatial reasoning as they try to use the geometric shapes in order to create pictures of familiar objects such as a boat or cat.

**Engaging.** Manipulatives can be an important tool in helping students to engage in mathematics. Moch (2001) described how she engaged students in a task that involved graphing by asking her students to make a bar chart of the candy, a manipulative, they would receive from trick or treating on Halloween. The students were engaged in mathematics by collecting candy, and as a class, making charts for the total candy collected then determining the mean, median, mode and range for the candy in their charts. Moyer (2001) studied teachers' perspectives on the role of manipulatives in the classroom. Moyer found that many teachers used manipulatives in their lessons in order to interest and engage students in doing mathematics. Ball (1992) explained that because manipulatives are real materials and physically present in front of students, they engage the students' senses, which can help promote mathematical activity on the part of the students.

***Physicality and Manipulability.*** Several authors have suggested that the physicality and manipulability of manipulatives can enhance student learning, although no explicit examples of this were provided. Hinzman (1997) explained that the use of manipulatives stimulates the senses of students as they are able to touch the manipulatives, move them about, rearrange them or see them in various patterns and groups. Resnick (1998) also argued that students can benefit from the use of physical objects that they can directly manipulate. Clements (2000) pointed out that computer manipulatives can provide a flexible way for students to change and rearrange a representation or various information. Although it may benefit students to rearrange and physically move manipulatives around, Moyer (2001) pointed out that the physicality of manipulatives does not carry the meaning of the mathematical ideas behind them. Students must reflect on their actions with the manipulatives in order to build meaning.

These studies have outlined different theories for the use of manipulatives. First manipulatives can help students' reason from concrete to abstract. Second, manipulatives can cue students to relate mathematics to their real-life experiences. Third, manipulatives can engage students in mathematical activity. Finally, manipulatives may be useful by allowing students to physically manipulate and rearrange concrete objects. These theories have discussed why manipulatives are useful to students, but they have not explicitly addressed how the manipulatives help students to develop mathematical understanding. There is still a question as to exactly what aspects of manipulative materials help students to build understanding, and to provide empirical evidence of such aspects (Manches, et al., 2010; Namukasa, et al., 2009). Is it their physicality or manipulability? Is it their role in engaging the senses, is it their ability to help students' reason abstractly or is it something more?

Given the literature on representations which suggests representations can help students build understanding in three main ways (by making connections between representations, making justifications and generalizations and relating mathematics to real life experiences) it is interesting to note that the literature on manipulatives has failed to explicitly address how manipulatives can help high school students build mathematical understanding. Manipulatives are often defined as a particular type of representation, but research has provided little or no clear evidence to connect how representations in general facilitate understanding and how manipulatives facilitate understanding.

**Manipulatives and the Pirie-Kieren Theory.** The Pirie-Kieren Theory provides a framework for the growth of mathematical understanding. I hypothesize that manipulatives may influence and promote the activity in which students participate within the levels of understanding, and prompt students to fold back and then move forward in the model. Within the inner levels of understanding, I expect that as the students begin to form images about the manipulatives, and that they will begin to recognize properties as they progress into the *property noticing* level of understanding. After describing and making connections between properties, I expect the students will make generalizations based on the properties they noticed about the manipulatives which will lead them into the *formalizing* level of understanding. After developing generalizations, students may extend these generalizations and progress into the *observing* level as they form theories based on the properties and generalizations they made about the manipulatives. After developing these theories, based on the manipulatives, I would expect the students to justify and prove their theories to be accurate or not, which is evidence of working in the *structuring* level of understanding.

The Pirie-Kieren Theory outlines the activities students may do within a given level of understanding, as well as describes how students move across various levels of understanding through folding back and moving forward. It will therefore provide an effective lens for this study as I analyze how manipulatives facilitate one group of high school students' mathematical understanding. By using the Pirie-Kieren Theory, I will be able to explicitly characterize how manipulatives are useful in helping students to develop mathematical understanding.

### **Section 3: Towers Task**

The Towers Task (Fig. 2) provides opportunities for students to use manipulatives when exploring various aspects of the task. In the following section, I will discuss the mathematical ideas and conceptions that can be elicited by the Towers Task. I will provide a summary of some students' explorations and previous research on the Towers Task.

#### **Towers Task**

- Build towers using linking cubes. Exactly two colors of linking cubes are available.
- Two towers are the same if they have the same colors, in order, from bottom to top.
- Towers can have cubes all of one color or cubes of each color.

A. How many different towers exactly four cubes high can be built from linking cubes when there are two colors available to choose from? How do you know that you have built all of the towers that are possible? How might you convince someone else that you have all the possibilities, that there are no more and no fewer?

B. What about towers "n" blocks high? Provide details on your investigations and problem solving approaches.

*Figure 2. Towers Task*

Students' solutions of the Towers Task have been extensively examined as part of a longitudinal study conducted by Rutgers University in the early 90's. The purpose of the longitudinal study was to examine the development of mathematical proof by young children in

the area of combinatorics. Much of the research focused on the types of justifications that one group of elementary students developed while completing the Towers Task during 3<sup>rd</sup>, 4<sup>th</sup> and 5<sup>th</sup> grade (Maher & Martino, 1996a, 1996b, 2000; Martino & Maher, 1999). Other studies focused specifically on students' use of the knowledge they gained through the Towers Task in order to develop ideas about Pascal's Triangle (Maher, 2005; Maher & Speiser, 1997).

One group of students began working on the Towers Task in the third grade. The first part of the task prompted the students to determine how many towers they could build that were four blocks high (four-high towers) given black and white Unifix cubes to choose from. Most students began the task by using a guess and check strategy to create new towers and to search for duplicates. When prompted to justify whether or not they had all possible towers, most of the students recognized they needed a more logical strategy in order to justify they had all of the four-high towers. When working on the Towers Task again in the fourth grade, the students developed a cases approach that would account for all of the towers that had a certain number of blocks of a particular color in each tower (Maher & Martino, 1996a; Martino & Maher, 1999). For example, students would categorize their towers into groups of towers with exactly zero white blocks, one white block, two white blocks, three white blocks and four white blocks, in order to justify they had all of the four-high towers within each case. This categorization of the towers helped the students to develop a convincing argument that they had all possible combinations of towers (Maher & Martino, 1996a).

In the fifth grade, the students worked on the second part of the Towers Task, which prompted them to think about how many towers they could build that were " $n$ " blocks high, given two colors to choose from. The most common type of justification developed for this portion of the task was an inductive argument. The students determined that the towers of a

given height could be created by adding either a white or black block onto every tower of a previous height, therefore the number of towers created a “doubling” pattern (Maher & Martino, 1996b, 2000). For example, there are eight three-high towers and to build four-high towers, one could add a white block and then a black block on top of each of the three-high towers. This would produce a total of 16 towers that are four blocks high. Thus, the number of four-high towers is double the number of three-high towers. Although the students did not develop formal notation, such as  $2^n$ , to describe how many towers they could build that were “ $n$ ” blocks high, they still developed an inductive argument in determining that the number of towers doubled given each subsequent tower height.

The authors of these previous studies were concerned with how the students’ justifications developed as they continued working on the Towers Task over an extended period of time. The students used Unifix cubes to build towers and arranged the towers in specific ways when using their cases and inductive approaches during justification. However, the authors did not explicitly state how the manipulatives were useful in helping the students to build understanding of the problem. It would seem that the ways the students arranged the towers in their cases approach, and the image of adding a block of each color to towers in their inductive approach would have helped them to develop their arguments, but the authors did not address how or why the manipulatives were useful to the students.

Maher and Speiser (1997) reported on an interview done with one of the students from the longitudinal study, Stephanie, who was in 8<sup>th</sup> grade at the time of the interview. The authors focused how Stephanie’s previous work on the Towers Task in elementary school helped her to reason about Pascal’s Triangle in relation to binomial coefficients. Stephanie asserted that the expansion of  $(a + b)^2$  was related to the two-high towers. Stephanie recognized that the term  $a^2$

could be thought of as one tower that had two blocks of one color (red), that  $2ab$  could represent two towers that have two different colored blocks each (red and yellow) and that  $b^2$  could be thought of as one tower with two yellow blocks. In a follow up interview, Stephanie made predictions about the coefficients of the terms in  $(a + b)^n$  by relating the entries in various rows of Pascal's Triangle to the number of  $n$ -high towers in each given case ( $k$  reds say, for  $k = 0, 1, 2, \dots, (n - 1), n$ ).

Maher (2005) provided another example of participants from the longitudinal study using their knowledge of the Towers Task in order to build meaning of Pascal's Triangle. In this study, Maher analyzed an interview done with four of the students who were in high school at the time of the interview. The students were challenged to give meaning to the addition rule used to build Pascal's Triangle. In response, the students showed that there was a correspondence between the terms in Pascal's Triangle and the Towers Task. The students related the notation  $\binom{n}{x}$  to denote the number of combinations of  $n$  things taken  $x$  at a time (in this case  $n$  represents the height of a tower and  $x$  represents the number of blocks of a particular color, say red, in the tower). They explained the structure of the addition rule as adding extra blocks of the  $x$  color onto a tower and wrote the equation  $\binom{n}{x} + \binom{n}{x+1} = \binom{n+1}{x+1}$  that represented the addition rule for Pascal's Triangle. One student, Ankur, explained the specific notation of  $\binom{3}{1} + \binom{3}{2} = \binom{4}{2}$  as being able to build the six four-high towers with two red cubes by adding a red cube to the three-high towers that have one red cube and by adding the other color block to the three-high towers that have two red blocks. So the resulting six towers are four blocks tall ( $n = 4$ ) and each have 2 red blocks ( $x = 2$ ) (Uptegrove & Maher, 2004).



The students in these two studies built on the knowledge they had gained by solving the Towers Task in order to develop understanding of the entries in the rows and of the addition rule in Pascal's Triangle. Stephanie related the entries in Pascal's triangle to the number of towers of a given height with a particular number of colored blocks in it. Through this experience, Stephanie built a rich understanding of what each term would mean in binomial expansions. The group of high school students was able to generalize the notation for the addition rule of Pascal's Triangle and to make sense of it by relating the notation to the process of building towers. Ankur provided a specific example of the addition rule by explaining how the rule would be applied to building four-high towers from the three-high towers. Although the students did not use manipulatives during either of these interviews, it stands to reason that the knowledge they had gained from completing the task with manipulatives when in elementary school provided them with a foundation to generalize and justify the patterns they observed in Pascal's Triangle.

### **Research Question**

The Pirie-Kieren Theory can provide an effective lens for how students build mathematical understanding of particular concepts. Mathematical representations, specifically manipulatives, can be a tool in helping students to build understanding as they help students to make mathematical connections and generalizations as well as model real world problems. Prior studies have addressed particular kinds of understanding that students have developed of combinatorics topics through the completion of the Towers Task. However, there is a lack of evidence about the relationship between manipulatives and the growth of high school students' mathematical understanding. The larger issue that has emerged from reviewing the literature on mathematical understanding is the question of how manipulatives can help students to develop their mathematical understanding. I will address this issue in my study by answering at the

following research question: how do manipulatives facilitate one group of high school students' understanding of combinatorics and Pascal's Triangle over time?

### **Chapter 3: Methodology**

This chapter outlines the case study I will use to answer my research question. I plan to use qualitative analysis of several data sources in order to determine how manipulatives are useful in helping students to build mathematical understanding. I will first describe the setting and participants of my study. Then I will outline how the data were collected and how analyzed the data in order to answer my research question. Finally, I will provide a justification of explicit analysis of three short episodes of data.

#### **Setting**

This case study is contained in a larger longitudinal study conducted as a two-semester collaboration between a large private university and a public high school. One university faculty member implemented a new model for the mathematics teacher education practicum course, wherein pre-service teachers enrolled in the course had opportunities to plan and teach several mathematics lessons to Algebra 2 and Geometry classes at the public high school during the 2009-2010 academic school year. The purpose of the larger study was to promote understanding of high school students' mathematical reasoning and sense making as well as pre-service teachers' development in their practicum course. I focus my case study on one group of high school students who were members of the Algebra 2 class participating in the larger case study. The study is an analysis of the students' collaboration on the Towers Task during three different after-school interviews during April 2010. Each interview lasted approximately one hour, for a total of three hours spent collaborating on the Towers Task.

#### **Participants**

The focus group includes Ashley, Jamie and Beth. At the beginning of the study, these students participated in a group interview designed to collect background information about their

grade, likes/dislikes about mathematics, previous mathematics courses, future plans for taking mathematics courses and their disposition toward mathematics. A summary of the students' backgrounds is provided in the table below (Table 1).

There were several reasons why these particular students were chosen as participants in this study. One reason was the girls represented a range of mathematical ability. Beth was fairly confident in her mathematical ability and performed well in the class. Ashley was less confident in her abilities, but she worked hard in class and on her homework. On the other hand, Jamie struggled in class, on her homework and on exams. Another reason why these students were picked was how they worked together as a group. During group time in their class, the girls were very talkative and willing to share their ideas. They were also very respectful of each other and were productive in attempting each task they were given. A third reason they were selected was that I had spent significant time recording their activity as a group during their Algebra 2 class periods and they were comfortable with me videotaping them. I assumed that using this particular group of girls would provide an opportunity to collect good video data.

At the time of this study all of the participants were high school sophomores who had taken Geometry during their freshman year. Ashley expressed how she liked to figure out “new things” when participating in mathematical problem solving, and that it made her feel “so good” when she gained understanding of mathematical concepts. She explained that she disliked fractions because she didn't understand the rules of the operations. The next mathematics course she planned on taking was College Math.

Jamie identified factoring as her favorite mathematical concept; she said that it was “the easiest thing all year.” Jamie enjoyed doing hands on mathematical activities and using manipulatives while problem solving. Her least favorite mathematical concept was also fractions.

Jamie planned on retaking Algebra 2 the next school year. Beth said she liked thinking about mathematical concepts in multiple ways, and participating in problem solving. She loved coming to “ah ha” moments in problem solving, where a mathematical concept began to make sense to her. She said the concepts of fractions and sector areas were frustrating to her. Beth planned on taking Pre-Calculus the next year.

Table 1

*Summary of students' background Information*

Student Name	Grade	Likes/Dislikes about Math	Previous Math class	Future Math class	Disposition toward Math
Ashley	10 <sup>th</sup>	Figuring out new concepts/Fractions	Geometry	College Math	Feels good when she understands concepts
Jamie	10 <sup>th</sup>	Factoring/Fractions	Geometry	Retaking Algebra 2	Enjoys tasks that require manipulatives
Beth	10 <sup>th</sup>	Thinking about concepts in multiple ways/Fractions & Sector Area	Geometry	Pre-Calculus	Loves “ah ha” moments

### **The Towers Task**

The students were given the Towers Task (Fig. 2), which was adapted by the researcher from Maher and Martino (1996a). In the task, students were asked to build towers using Unifix Cubes. There were exactly two colors of cubes, red and blue. In the directions, the students were told that two towers are the same if they have the same color of blocks from top to bottom. In

part A of the task, the students were asked how many different towers they could build that were exactly four blocks high, given two colors to choose from. They were encouraged to explain how they knew they had built all of the towers that were possible. In part B of the task, the students were asked to determine how many towers they could build that were “ $n$ ” blocks high. In both parts of the problem, the students were required to provide justification for their solutions. The students had no previous experience with the Towers Task, and were not given any prior instruction before starting the task other than the author providing an example of two towers that were the same. I chose to introduce this task to the students because of the opportunity it would provide the students to use manipulatives as they completed the task, and because of the potential to elicit ideas about combinations and Pascal’s Triangle.

### **Data Collection**

The author videotaped all interviews, with each interview lasting about an hour, for an approximate total of three hours of video data. During each interview the camera was positioned in a way to record what the students said and did during each interview. Facial expressions and gestures made by the students were recorded as often as possible in order to provide more information for interpretation of the students’ understanding. While collecting data, the author was considered a “participant as observer” (Preissle & Grant, 2004) with the role of video recording and observing each interview, as well as facilitating the students’ completion of the task.

After each interview, I viewed the video data and took notes about the students’ work in order to develop a better understanding of the students’ progress with the task. I designed questions, for the subsequent interview, that would help me understand their mathematical ideas and help them clarify their justifications, as well as how to continue progressing with the solution

strategies they had developed. As part of the completion of the task, the students were encouraged to write out their solutions and solution strategies on paper. These documents served along with the video as sources of data.

### **Transcripts**

I transcribed and annotated each interview in its entirety. Annotations bracketed within the transcript consisted of gestures the students used, any actions the students demonstrated, the towers they built and how the towers were organized, as well as clarifications of pronouns used such as *this, it, that, those* etc. As the students' often referred to their work throughout their discussion of the task, I created replications of their work to clarify and annotate pieces of the transcripts. Research team members are going to verified each of the transcripts to make sure the transcripts and annotations were accurate.

### **Data Analysis**

My research question aims to address how manipulatives facilitate the growth of mathematical understanding of combinations and Pascal's Triangle. Since I am looking for connections between two main components of my study, manipulatives and mathematical understanding, I have used both video data and student work to analyze both components of my research question. I explain my techniques further below.

I began analysis by viewing each interview and made note of critical episodes of the interviews. These critical episodes consisted of times when the students were using the manipulatives in powerful ways and when they were progressing with their ideas. After taking note of these critical episodes, I developed Pirie-Kieren codes that represented the different levels of understanding in the Pirie-Kieren Theory. I performed line-by-line coding of each interview using the Pirie-Kieren codes. After completing this coding, I analyzed the critical

episodes within each interview, paying attention to the ways in which the students were using the manipulatives. I then developed several open codes that corresponded with the students' use of the manipulatives as well as codes to characterize the mathematics the students developed based on the manipulatives. After I developed each of the open codes pertaining to manipulatives, I performed line-by-line coding of each interview using these codes.

After coding each interview for manipulatives and levels of understanding, I focused on finding patterns between the two sets of codes. The connections I made with how the students' use of manipulatives corresponded with particular levels of understanding helped me to refine the ways in which the students used the manipulatives within each level of understanding. These connections prompted me to adjust several instances of the Pirie-Kieren codes I had. For instance, many times throughout the interviews, the students used the manipulatives to help them explain or justify their ideas, which corresponded with working in *observing* level of understanding. After making this connection, I went back through each time the students explained or justified their ideas to assure that the Pirie-Kieren code of that line was accurate.

After adjusting the Pirie-Kieren codes, I focused on each of the critical episodes that I felt were compelling. I analyzed each episode to make sense of the mathematics that was being developed as well as how the manipulatives were facilitating understanding. I created justifications for why I assigned pieces of transcript particular codes, which helped me build a nuanced definition of each code. While paying attention to the mathematical story that unfolded throughout the interviews, as well as the justifications of the coding, I identified common themes and patterns in how the manipulatives facilitated the growth of these students' mathematical understanding.



## Pirie-Kieren Codes

Throughout my analysis I used codes to characterize the levels of understanding that the students were working in (see Table 2). I identified the *primitive knowing* (PK) level whenever the students referenced prior mathematics and tried to connect that piece of mathematics to the Towers Task. I coded *image making* (IM) whenever the girls developed initial images and conceptions about the new concepts they were dealing with in the Towers Task as well as its extension to Pascal's Triangle. *Image having* (IH) was coded each time the students extended the initial images they had developed without having to reference the action that brought about the initial image. For instance, I coded *image making* when the girls started to randomly build four high tall towers, but then coded *image having* when the girls talked about a particular tower without having to build it. *Property noticing* (PN) was coded each time the girls referenced a property they saw within the towers, within a particular arrangement of the towers, or within a theory they developed about the concepts related to the Towers Task or Pascal's Triangle. I identified the *formalizing* (F) level whenever the girls developed a generalization based off of the properties they had noticed. When the girls extended the generalizations they had developed and logically organized their ideas to create theories, I coded them within the *observing* (O) level of understanding. The *structuring* (S) level was coded any time the girls justified or explained their theories in an attempt to validate whether their theories were accurate or not. Finally, I coded the *inventising* (I) level whenever the girls had fully justified their theories about a particular piece of the Tower's Task or Pascal's Triangle and then moved on to work on a different part of the task.

Table 2

*Codes for Pirie-Kieren levels of understanding*

Pirie-Kieren Level of Understanding	Code
Primitive Knowing	PK
Image Making	IM
Image Having	IH
Property Noticing	PN
Formalizing	F
Observing	O
Structuring	S
Inventising	I

### **Manipulative Codes**

There were several different manipulative codes I developed in order to help me make sense of how the girls were using the manipulatives to help them build understanding throughout the interviews (see Tables 3 and 4). I separated the codes into different categories, the first being codes pertaining to how the girls were using the manipulatives throughout the task and second being codes that related to particular mathematics developed through use of the manipulatives.

Throughout the interviews, there were many times when the girls physically handled the manipulatives in some way or when they focused on physical properties of the manipulatives. During the Towers Task and its extension to Pascal's Triangle, the girls oftentimes physically built the towers or built a particular arrangement of the towers. Each time an action like this occurred, I coded it as *building* (B) as well as the times when they referenced what they would do to build a tower or arrangement even if they were not physically building it at that moment. I

coded *arrangement* (AR) any time the girls placed the towers into a specific arrangement, referenced their arrangement or explained an arrangement. I developed a *properties* (PRT) code which accounted for the times when the girls referenced or described properties they noticed within or between towers as well as within arrangements of towers or theories they had developed based on the towers.

There were also many times when the manipulatives facilitated the girls' solution strategies as well as their ideas and explanations pertaining to the task. Whenever the manipulatives prompted the girls to follow some particular idea or piece of reasoning, I coded this as a *prompt* (PMT). There were several times throughout each interview when the girls would change their solution strategy or their line of thinking, each time this change was prompted by the manipulatives, I coded it as *change* (CH). Throughout the interviews, the girls provided many explanations and pieces of reasoning that were prompted by the manipulatives or the theories that they developed from the manipulatives. Any time the girls provided an explanation such as this I coded it as *explanation* (EX) and any time they provided a piece of reasoning I coded it as *reasoning* (R). Each time the girls clarified their ideas or understanding, I coded this as *clarification* (CL). During the latter part of the first interview and throughout the second and third interviews, there were times when the girls abstracted based on their ideas and theories from the manipulatives. I coded these instances as *abstracting* (AB). Oftentimes the girls described and explained the connections they saw within particular towers or arrangements, as well as the connections between towers, arrangements and theories. Each time the girls saw a connection within towers or arrangements, I coded this as *connections within* (CW) and each time they saw a connection between towers or arrangements, I coded it as *connections between* (CB). In the second and third interviews, the girls made several connections between different

theories that they developed about the manipulatives, so I coded these instances as *connections between theories* (CBT).

There were several different manipulatives codes I developed to characterize the different pieces of mathematics that were elicited by the students' use of the manipulatives. The girls developed an inductive argument to prove how the different heights of towers were related to each other. This argument was based on a Doubling Theory (explained in more detail later) that the girls developed during the first interview. Each time the girls referenced this inductive argument or Doubling Theory, I coded it as *induction/doubling* (ID). Related to this inductive argument, the girls also developed an idea as to how the number of possibilities (or towers) of a particular height were connected to the number of a possibilities of another given height. I coded instances where they discussed this as *possibilities* (PO). Throughout the interviews, there were times when the girls developed algebraic expressions to define the patterns they were seeing based on their arrangement of the manipulatives. I coded these episodes as *algebraic expression* (AE). In the second and third interviews, the girls focused on how the towers related to the additional rule and number entries in Pascal's Triangle. I coded the development of the addition rule as *Pascal addition rule* (PAR) and the development of the number entries as *Pascal numbers* (PNU).

Table 3

*Codes for students' use of manipulatives*

Students' use of the manipulatives	Code
Building	B
Arrangement	AR
Properties	PRT
Prompt	PMT

Change	CH
Explanation	EX
Reasoning	R
Clarification	CL
Abstracting	AB
Connections Within	CW
Connections Between	CB
Connection Between Theories	CBT

Table 4

*Codes for mathematics elicited by manipulatives*

Mathematics elicited by manipulatives	Code
Induction/Doubling	ID
Possibilities	PO
Algebraic Expression	AE
Pascal Addition Rule	PAR
Pascal Numbers	PNU

### **Justification of Codes**

In my initial analysis of the data, I completed line-by-line coding of each transcript, starting first with the Pirie-Kieren codes followed by the manipulative codes. In this section, I will describe my justification for coding three small pieces of transcript, one piece from each of the interviews. Before I provide the justification for these pieces of transcript, I will clarify some notation that I will use throughout the analysis, as well my decisions for my Pirie-Kieren coding. I have created abbreviations that describe particular towers the girls built or discovered. For example, the tower with red blocks as the top two blocks and blue blocks as the bottom two

blocks is called RRBB. The tower with blue as the top and third blocks and red as the second and fourth blocks is called BRBR etc. Also, I have created abbreviations that describe a group of towers with a particular number of colored blocks. For example, towers with 3 blue blocks and 1 red block, are called 3B1R towers. The towers with 2 blue blocks and 2 red blocks are called 2B2R etc.

As discussed earlier, the Pirie-Kieren model consists of eight levels of understanding, which are represented as eight embedded circles. Pirie and Kieren (1992) emphasized that each level of understanding contains all previous levels, and is embedded in all succeeding levels of understanding. This would imply that students may be able to work within multiple levels at a time, such *property noticing* and *formalizing*, as *property noticing* is embedded in the succeeding levels of understanding. However, there is a lack of detailed explanation of how students can work within multiple levels of understanding at the same time, and what that may look like. Throughout my analysis, there were many times in which the students seemed to be working within multiple levels of understanding simultaneously. Because the literature on the Pirie-Kieren Theory failed to explicitly address this issue, I proceeded to characterize why and how the students could be working within multiple levels, based on the evidence I saw within the data. These justifications will be explained in more detailed within the analysis and conclusions of this thesis, but the reader can now assume that students may be able to work within multiple levels of understanding at the same time.

The first piece of transcript (Table 5) occurred several minutes after the girls began the Towers Task and describes a particular arrangement of the towers that Ashley suggested the group place the towers in. Prior to Ashley's suggestion, the group had been randomly building and placing the towers on their desks.

Table 5

*Transcript and codes*

Transcript	Pirie-Kieren Codes	Manipulative Codes
Ashley: Ah k, these two are opposite [picks up RRBB and BBRR]	PN, F	B, AR, PRT, PMT, CH, R, CW
Jamie: Yeah put them with their opposites	PN, F	AR, PRT, R, CW
Ashley: And then these two [BRRB & RBBR], these two [RBRB & BRBR], these two [RBBB & BRRR], these two [BRBB & RBRR], these two [BBRB & RRBR], these two [BBBR & RRRB], and these two [BBBB & RRRR].	PN, F	B, AR, PRT, R, CW
Stacie: Okay, can you, can you tell me a little bit about the way that you've organized them there [the towers].		CL
Ashley: We did opposites	PN, F	AR, EX
Stacie: Okay, so meaning?		CL
Ashley: Like, two blues on top and then two reds [points to BBRR]. And then two reds on top and then two blues on the bottom [points to RRBB] And same as this like one red on top [points to RBBR] a blue on top [points to BRRB], and then the same color on the bottom [points to the top red and bottom red of RBBR]	PN, O	AR, PRT, EX

In this short episode, Ashley reasoned (R) about a connection she saw within (CW) the properties (PRT) of the given towers, which prompted (PMT) her to build (B) and arrangement (AR) where each tower was placed with its opposite pair. This was a change (CH) in strategy as before her recognition of this connection, the group had no logical way of organizing the towers. Ashley is working within the *property noticing* and *formalizing* levels of understanding here as

she noticed the properties within the towers and formed a generalization that they could pair the towers up based on the opposite connection she noticed. After I asked Ashley to clarify this new organization of the towers, she went on to explain (EX) the properties (PRT) of their “opposite” arrangement (AR) and provided examples of what she meant by “opposites.” After grouping the towers by opposites, she then elaborated on the generalization and formed it into a theory by explaining the properties she was seeing and providing examples of opposite pairs, which is evidence of working within the *observing* level.

At the end of the first interview, the girls had discovered that the expression  $2^n$  would give them the number of towers of any particular height,  $n$ , given two colors of blocks to choose from. Prior to the second piece of transcript, the girls had discovered that when they arranged their towers by case (Maher & Martino, 1996a; Martino & Maher, 1999), focusing on how many blue blocks were in each tower and then counted the number of towers in each group, these counts corresponded with the rows in Pascal’s Triangle. They had filled in the first four rows of Pascal’s Triangle and were in the process of determining what counts would go in the fifth row when the following exchange took place.

Table 6

*Transcript and codes*

Transcript	Pirie-Kieren Codes	Manipulative Codes
Beth: One, five, then five one, but what was in the middle? [has written out on her paper 1 5 5 1] This should be three numbers right? [points to the open middle of the 5th row] Or is it? Let me write it out. There's one [writes 5-1], four there would be five [writes 4-5]	PN, O	AR, PRT, R, CL, CB, PNU,
Stacie: K Beth, what are you doing here?		CL



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Beth: I'm trying to figure out, if you use, like if there were five blue cubes there would be one tower [points to 5-1], four blue cubes there would be five towers [points to 4-5]

Beth continued until she had the following list of tower counts

5 – 1 [5 blue blocks, 1 tower]

PN, O

AR, PRT, AB, EX,  
CB, AE, PNU

4 – 5 [4 blue blocks, 5 towers]

3 –

2 –

1 – 5 [1 blue block, 5 towers]

0 – 1 [0 blue blocks, 1 tower]

Ashley: They, they'll all add up to thirty-two though

O

R, CBT, PNU

Beth: Yeah

Stacie: Why?

Ashley: Because five, two to the fifth is thirty-two

O, S

AB, EX, AE, CBT

Beth: So these two have to add to twenty [points to 3- and 2- ]. But are they ten and ten?

PN, O

AR, R, CL, PNU

---

In this short episode, Beth reasoned (R) about how to fill out the numbers in the fifth row of Pascal's Triangle (PNU) and gave meaning to those numbers by making the connection between (CB) the numbers and the way they had arranged (AR) their towers by case. By focusing on the properties of the counts of towers in each particular case, Beth created a numerical list (AE) that helped her organize the numbers in the fifth row of Pascal's Triangle. Beth recognized properties of their cases arrangement and developed a theory as to how to apply their cases arrangement to determining the numbers in the 5<sup>th</sup> row of Pascal's Triangle, which is indicative of the *property noticing* and *observing* levels of understanding. After Beth created this list, Ashley made a connection between how Beth's theory of finding the numbers in the

fifth row to their abstract theory of finding the number of towers of any particular height,  $2^n$  (CBT). Ashley explained (EX) and justified that there would be 32 five-high towers because  $2^5 = 32$ . This justification of their abstract theory helped move her into the *structuring* level of understanding. In Beth's last exchange, she was still trying to apply their cases arrangement to the last counts in the fifth row of the triangle (PNU). She reasoned (R) whether or not there would be ten towers in each case and her clarifying question (CL) led the group to build all of the five high towers in order to justify that their counts were correct. Although Beth was still developing how their cases arrangement applied to their theory of filling out the rows of Pascal's Triangle, she could not yet justify the connection, therefore she had not moved into the *structuring* level of understanding at that point and was still working in the *property noticing* and *observing* levels of understanding.

In the first interview, the girls had developed a "Doubling" Theory, similar to the doubling pattern described in, (Maher & Martino, 1996b, 2000) in order to prove they could create all of the towers of a particular height by placing either a red block or blue block on top of each tower in the previous height. At the beginning of the third interview, I prompted the girls to determine why the addition rule in Pascal's Triangle works and they spent the first ten minutes of the interview suggesting ideas as to how their Doubling Theory might be related to the addition rule, but they could not justify their ideas. The girls had written out Pascal's Triangle on a piece of paper and in the following exchange, related their Doubling Theory to the addition rule in Pascal's Triangle.

Table 7

*Transcript and codes*

Transcript	Pirie-Kieren Codes	Manipulative Codes
Stacie: Okay, so where's this tower [RRR] going to be in Pascal's Triangle? If we were gonna place it in Pascal's Triangle, where would it go?		CL
Beth: Right there [placed RRR on the left "1" in the third row of Pascal's Triangle]	PN, F	B, AR, PRT, R, CB, PNU
Stacie: Okay, so now what did you do to that tower [RRR] to get it to be four tall? Four blocks tall		CL
Beth: You'd add a red or a blue	PN, O	B, AR, PRT, EX, ID, CW
Stacie: Okay so if you add a red block to that tower... [Beth then made RRRR]		CL
Beth: Then it would go right there [put RRRR on the left "1" in the fourth row]	PN, F	B, AR, PRT, R, CBT, PNU, PAR
Stacie: Okay and what happens if you add a blue onto that tower?		CL
Beth: Then it would go in here. [made BRRR and puts in on the left "3" in the fourth row].	PN, F	B, AR, PRT, R, CBT, PNU, PAR
Beth: Okay I think this might work, because you can take . . . we'll start with the two. . . [built RB and placed it on the "2" in the second row] so if I were to take this tower [RB], you could add either a red one or a blue [made RRB and BRB], and then um, this tower [RRB] would then go here [on top of the left "3" in the third row] because it fits having only one blue so it would go on top of this three in that kind of group. But since this [BRB] still has two blues, it goes here [placed BRB on the right "3" in the third row].	PN, O, S	B, AR, PRT, EX, CBT, PNU, PAR

In this episode, Beth reasoned (R) about how she could create a physical representation of Pascal's Triangle using the theory they had developed in the second interview about what the numbers within Pascal's Triangle mean (PNU) in terms of the towers. This physical representation helped Beth to make connections between (CBT) their Doubling Theory and their theory about the meaning of the numbers in each row of the triangle. The connections Beth made between these theories helped her to develop an initial understanding of the addition rule in Pascal's Triangle (PAR) and how the addition rule could be physically modeled with the towers.

Beth recalled the connection between (CB) the numbers in Pascal's Triangle and how those numbers related to the properties (PRT) of colored blocks within each tower. Beth then arranged (AR) the towers in a very specific way by physically placing (B) the RRR tower where it would go on their numerical representation of Pascal's Triangle. Then Beth connected the Doubling Theory to their theory of what the numbers within Pascal's Triangle mean when she built RRRR and BRRR and placed them accordingly in the triangle. By making the connection between those two theories (CBT) Beth created a new theory about how they could model the addition rule in Pascal's Triangle with the towers. Beth explained (EX) and justified this theory by providing an example creating RRB and BRB by placing a red block and then a blue block on RB and then showing where RRB and BRB would be placed in the next row of Pascal's Triangle.

Throughout this small episode, Beth moved from the *formalizing* level of understanding up through the *structuring* level of understanding. Throughout the exchange, she was continually working the *property noticing* level of understanding because she often referenced properties of towers and theories. When I prompted her to determine where RRR would be physically placed in Pascal's Triangle, Beth made the generalization that the towers could be physically placed in

the Triangle, and that they would be placed on the number that corresponded to the number of towers of a particular height and color case. When I asked her how RRR could be used to create four-high towers, Beth referenced back to the property they had developed in their Doubling Theory, that a red block and a blue block would have to be added on top of RRR. At this point, Beth had moved into the *observing* level of understanding because she was referencing the Doubling Theory. Beth moved back into the *formalizing* level of understanding when she placed RRRR and BRRR into Pascal's Triangle because she could not yet explain this generalization of physically placing the towers into Pascal's Triangle. Once Beth stated, "I think this might work." she was then thinking of this idea as a theory that she would then eventually prove accurate. This moved her into the *observing* level of understanding. Beth moved into the *structuring* level of understanding when she provided another example using RB and explained how her theory worked and why the towers were placed in that particular way according to the numbers in Pascal's Triangle.

It is interesting to note that in this episode, Beth worked in the *observing* level of understanding two different times because she was referencing and working with two different theories. The first instance of *observing* happened when Beth referenced back to the Doubling Theory. On the other hand, the second instance of *observing* occurred when Beth went on to explain how to use the Doubling Theory to model the addition rule in Pascal's Triangle.

## Chapter 4: Data and Analysis

In the data and analysis chapter, I will provide a narrative of compelling episodes of student work throughout the three interviews that illustrate how manipulatives facilitate the growth of mathematical understanding. The narrative outlines how students used the manipulatives to build understanding of the Towers Task during the first interview, followed by how the students used manipulatives to build understanding of Pascal's Triangle during the second and third interviews.

For the purpose of this study, I have defined manipulatives as a physical representation that can be manipulated by students to help justify and clarify their mathematical ideas. In this study, the Unifix Cubes, as well as the individual towers the girls created while engaged in the Towers Task and exploration of Pascal's Triangle are considered to be manipulatives. Unifix Cubes are an example of manipulatives; they are physical objects or representations that the girls used in order to create individual towers. These towers, in turn, became manipulatives themselves as the girls manipulated the towers into various arrangements and used the towers in order to justify and clarify their ideas throughout each interview. Throughout the rest of this thesis, reference to Unifix Cubes or towers can be considered reference to manipulatives.

### Applying Towers to Combinatorics

**Initial Student Work.** Initial work on the Towers Task included the girls' interpretation of the task and their initial ideas as to how they could build all of the four-high towers, given two colors of blocks to choose from, red or blue. As with the students in Maher and Martino (1996a), the girls in this study began the task using a guess and check strategy by building towers randomly, then comparing the towers they had built with their group to determine if any duplicate towers had been built. Although the girls were not organizing the towers in any

particular way, they were describing the towers they built in terms of the color of blocks in each tower, such as “red on top and one on bottom” (RBBR). Also, the girls noticed that particular colored blocks could “move” through in each spot of the tower. For instance, Ashley explained that you could have a blue block on top (BRRR) and that the blue block could move down to the second spot (RBRR), then the third spot (RRBR) and finally down to the fourth spot (RRRB).

After the group had built all 16 of the four high towers, Ashley suggested they should organize the manipulatives and the group developed two different arrangements of towers. The first arrangement the girls developed was a cases approach similar to Maher and Martino (1996a) where all of the towers with no blue blocks, exactly one blue block, exactly two blue blocks, exactly three blue blocks and exactly four blue blocks were grouped together. For the second arrangement of the towers, the girls placed each tower with its “opposite” tower. For example, Ashley placed RBBB and BRRR (Fig. 3) together because when comparing the two towers, there were different colored blocks in each spot (red in the top spot vs. blue on the top spot, blue in the second spot vs. red in the second spot etc.). After the girls developed these two arrangements of the four-high towers, I asked them how they could justify that they had all of the four-high towers. Beth reasoned that it would be easier for them to verify they had all the towers using their cases approach.

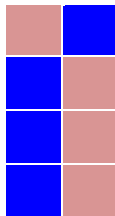


Figure 3: A pair of opposite towers, RBBB and BRRR

In the students’ initial work with the Towers Task, their understanding progressed from the *image having* level of understanding to the *formalizing* level as they noticed properties within

the towers, and placed the towers into specific arrangements. When the girls randomly built the towers without any type of organization, they were within the *image having* level. Then they noticed properties within the towers, as they started to describe the towers in terms of the color of blocks and pointed out that a particular color of block could occupy each spot in the tower. Noticing the properties within the specific towers moved them into the *property noticing* level. The girls progressed into the *formalizing* level of understanding when Ashley then suggested they should organize the towers, which prompted the girls to make connections about the properties they were seeing with the color of blocks in the towers. The connections made, based on the properties of the towers, helped them develop generalizations about how the towers should be grouped either by case, or in opposite pairs.

**Cases Theory.** Going back to their cases approach, the girls placed ten of the towers in the following arrangement (Fig 4.), which accounted for the cases of zero blue blocks, exactly 1 blue block, exactly 3 blue blocks and exactly four blue blocks. Beth explained an important property of this arrangement when she stated that the red block was “going down” through each spot in the 3B1R case and similarly, that the blue block was “going down” through each spot in the 1B3R case. Beth justified they had all of the 3B1R towers in their arrangement because there was no other spot within the tower to place the blue block, therefore they could not make any more towers within that case. She used similar reasoning to argue that they had all of the 1B3R towers in their arrangement and that there could only be one BBBB and RRRR towers.



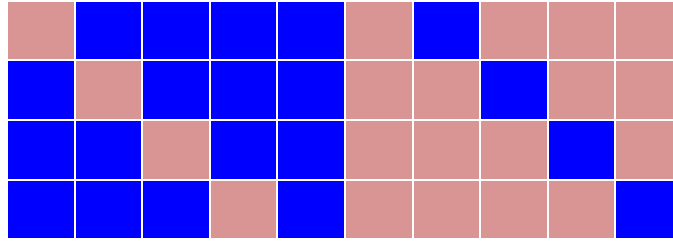


Figure 4: The group’s initial “cases” arrangement

After justifying those four cases, the girls moved on to proving they had all of the 2B2R towers by using a similar argument as above. Beth and Ashley placed the following towers in a line (Fig 5.) and Ashley explained that with three of the towers, there was one red block in the bottom spot with the second red block moving through each other spot in the tower (left three towers in Fig. 5). Ashley also explained that with the other three towers, there was one blue block in the bottom spot with the second blue block moving through each spot in the tower (right three towers in Fig. 5). They related their argument that a particular colored block could move through each tower in the 3B1R and 1B3R cases to prove they had all the towers in this 2B2R case. They recognized they would have to hold a color (red) constant at the bottom of the tower, while the other red block “moved” through the other spots.

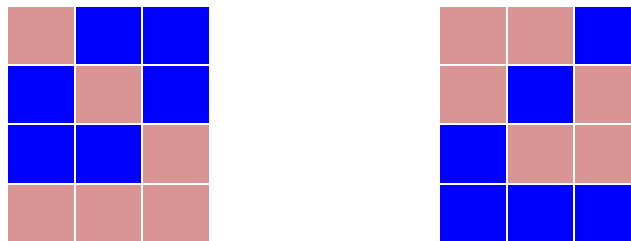


Figure 5: Arrangement of the 2R2B case

In justifying they had all sixteen of the four high towers, the girls progressed from the *formalizing* level of understanding to the *inventising* level of understanding by developing a theory as to how the towers should be arranged, and then noticing properties about their arrangement which helped them prove their theory was accurate and helped them justify they had

all of the four-high towers. When asked to justify how they had all sixteen towers, the girls decided they could more easily justify their ideas with the cases approach rather than the opposites approach. At this point they were in the *formalizing* level of understanding because they viewed the cases approach as a generalization. They had yet to think of it as a theory because they had not explained every case, or why the approach would help them justify they had all of the towers in a particular case. Once the girls arranged the towers as in Figure 4 and started explaining the properties of the arrangement, such as the red block “moving down” in the 3B1R case, they began to view the cases approach as a theory, and were working within the *property noticing* and *observing* levels. Their progression into the *structuring* level occurred when they explained and justified their Cases Theory and that they had accounted for every tower in each case. The theory building that the girls did based off of their cases arrangement helped them to justify they had all the four-high towers and as they had completed this portion of the task they moved into the *inventising* level of understanding and were ready to move on to a different part of the task.

The manipulatives in this episode were significant in helping the girls to progress through the different levels of understanding. The girls had to focus on the properties of the towers and make connections between the towers within each case in order to move past the generalization of grouping the towers together by color, toward arranging the towers in the specific ways they did in Figures 4 and 5. The arrangements of the towers prompted the girls to begin to think of their cases approach as a theory. The girls then reasoned about and explained the properties of the Cases Theory, which helped them justify they had all of the towers within each case.

**Doubling Theory.** After the group justified they had all of the four-high towers, given two different colored blocks to choose from, I asked them whether the number of three-high

towers would be larger or smaller. The girls immediately said there would be a fewer number of three-high towers, and Beth reasoned they could create three-high towers by removing the top block from each of their four-high towers and that some of the remaining three-high towers would be the same. She elaborated on this generalization by providing an example and explaining that if they were to take the top block off of RBBB and BBBB, the resulting tower, BBB, would be the same for both original four-high towers (Fig. 6). The girls proceeded to pair up the towers, which would create the same three-high tower if the top block was removed (RBBB & BBBB; RRRR & BRRR; RBBR & BBBR; BBRB & RBRB; BRBB & RRBB; BBRR & RBRR; BRBR & RRBR; BRRB & RRRB). From this grouping of the towers, they removed one tower from each pair, so that the remaining eight towers (Fig. 7) represented the number of three-high towers.

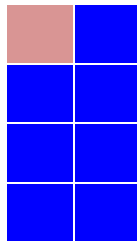


Figure 6: Removing the top block off of RBBB and BBBB would produce BBB.

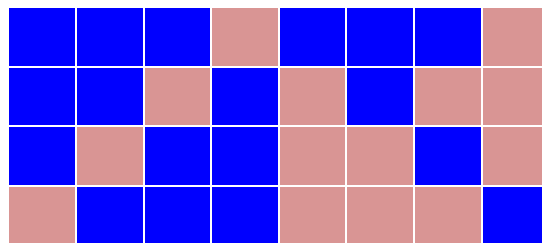


Figure 7: The group’s representation of the eight “three-high towers”

Beth’s generalization that removing the top block off of a four-high tower to make a three-high tower was based from a property she noticed within the towers, that she could see a

three-high tower within each four-high tower. As Beth described this generalization, she was working in the *property noticing* and *formalizing* levels of understanding. By elaborating on this generalization and explaining a specific example with RBBB and BBBB, Beth began to think of this generalization as a theory they could use to create all of the three-high towers, and helped her to progress into the *observing* level of understanding. In this case, Beth's *property noticing* helped her to see connections between the three-high and four-high towers and the connections she saw between the properties of particular four-high towers was what prompted the movement from the *formalizing* level to the *observing* level of understanding.

The girls were not sure if this strategy of producing three high towers would help them to prove they had all of the towers, so they decided to build the three high towers from scratch instead of trying to represent them using the eight four high towers in Figure 7. The girls could not use their theory of how the four-high and three-high towers were related in order to prove they had all of the three high towers, so they had to fold back to the *image making* level of understanding in order to build the three-high towers from scratch. After building all eight of the three high towers, the girls used their Cases Theory to create and arrange the three high towers in the following way (Fig. 8), as well as to justify that they had all of the three-high towers. Arranging their towers by case helped the girls return to the *observing* level and justifying they had all eight towers based on the properties of the Cases Theory helped them to progress into the *structuring* level of understanding

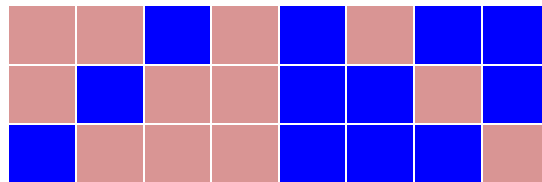
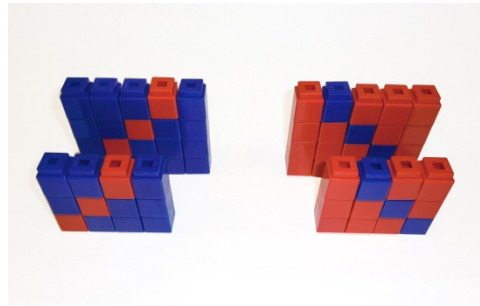


Figure 8: The group's arrangement of the three high towers

After they had built and arranged all of the three-high towers, I asked the girls how they could get from three-high towers to four-high towers. Beth reasoned that they should “match” the towers up and proceed to place the three-high towers in a line in front of their corresponding four-high towers (Fig. 9).



*Figure 9: “Matching up” the 3-high and four-high towers*

After Beth placed the towers in this arrangement (Fig. 9), the girls focused on a property they had noticed in these particular cases of towers, that the blue block moves up each spot in the 1B3R and 1B2R towers, and similarly that the red block moves up each spot in the 3B1R and 2B1R towers. They described this property as the “pyramid pattern” and Beth explained that as they moved to the next height of tower that it was, “like a pyramid, it could just keep on going.” When I asked what she meant by “keep on going” Jamie helped clarify by explaining that they would be “adding an extra cube” onto each tower in these cases to get to the next height of tower. Beth and Ashley provided the following examples:

Beth: Yeah so if we were to keep going, it would be... Four blues and a red [puts her finger on top of BBBB] and five reds and a blue [puts her finger on top of RRRR]

Ashley: It would just continue our [pyramid] pattern up. Like for this one, you would add another red [put a red block on top of BBBB] to make it a five [high] tower. And then if we had like another [builds BBBBB and puts it right next to RBBBB] five tower then we would add like our sixth [puts a red block on top of BBBBB] (Fig 10).



Figure 10: Ashley's arrangement of the towers

When prompted, the girls affirmed that their theory to get from towers of one height to the next height was to add a block on top of the lower height. The girls then spent the next six minutes explaining this pyramid theory, but they struggled to apply the theory to form all of the four-high towers from the three-high towers.

Within this part of the interview, the girls worked within the *property noticing*, *formalizing*, *observing* and *structuring* levels of understanding. Beth developed a generalization, based on the property that the three-high towers could be “matched up” in some way with the four-high towers, which is evidence of working in the *property noticing* and *formalizing* levels of understanding. Before developing a theory as to why the towers were matched up and arranged in that particular way, the girls identified a property they noticed within that arrangement of the towers, that the off colored block could move through each spot of a tower (Fig. 9). Although this particular property led to their justification of the pyramid theory, and that they could add a block on top of a lower height of tower to get to the next height, the girls were still unable to make a justification as to how they could use the three-high towers to build *all* of the four-high towers. The girls developed their pyramid theory based on the particular arrangement of the towers in Figure 10 and the properties they noticed within that arrangement. They were working in the *property noticing* and *observing* levels of understanding while creating their pyramid

theory, and then moved into the *structuring* level of understanding when they justified the theory by using the towers to guide their justifications.

Since the girls were struggling to progress with their pyramid theory, I prompted the girls to return to the *formalizing* level of understanding that took place when Beth reasoned that they could “match up” up the three-high towers with some of the four-high towers (Fig. 9). I asked the girls to explain their arrangement of the towers, and specifically why they placed RRB in front of RRRB.

Ashley: Because if we chop off the four [the top red block], it would be the same [then you would have two towers that are both RRB].

Beth: So it follows the same pattern, the top is just not the same. [the bottom three blocks of each four high tower corresponded to one specific three high tower]

After they gave their explanations, I pointed out they only had ten of the four-high towers in the arrangement, and asked them where the other six four-high towers would go in the arrangement. The girls placed the remaining six four-high towers on the other side of the three-high towers by matching the remaining four-high towers with their corresponding three high towers (Fig. 11 and 12).

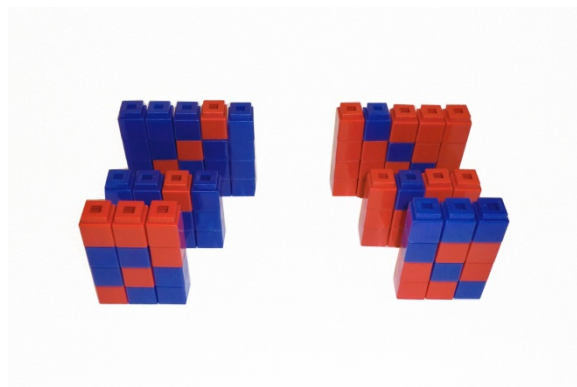
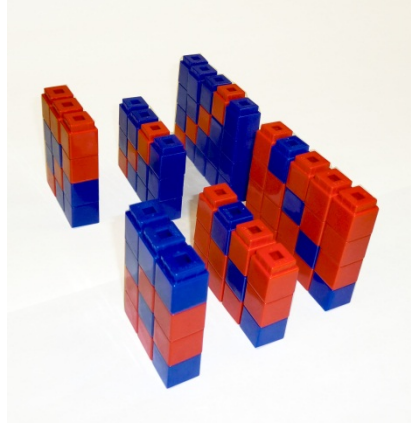


Figure 11: Matching the remaining four-high towers



*Figure 12: Matching the remaining four-high towers.*

Once the girls had placed the remaining four-high towers into their arrangement, I asked them what they had done to each of the three-high towers to get up to the four-high towers. Immediately after I asked this question, Ashley explained, “We either added a blue or a red [block on top of the three high towers].” Beth elaborated on Ashley’s answer by explaining that there were “two possibilities for each” three-high tower, meaning that two four-high towers could be created from every three-high tower by adding either a red or blue block on top of the three-high tower.

Here, the girls made connections between the properties of the three-high and four-high towers and then built an arrangement (Fig. 11 and 12) that helped them to explain they could use the three-high towers to create the four-high towers by adding either a red or a blue block on top of each three-high tower. It was the particular arrangement that helped them to see the recursive nature of the towers. Before placing the towers into the “doubling” arrangement, the girls were focused on trying to show that the towers were related because of the pyramid pattern, but this pyramid pattern did not help them justify how they could create *all* of the four-high towers from the three-high towers. Once they placed the towers into the doubling arrangement (Fig. 11 and



12), they explained and justified the recursive nature of the towers, that for every three-high tower, there would be two more four-high towers in the next group.

In this episode, the girls were working in the *property noticing*, *observing* and *structuring* levels of understanding. They created their doubling arrangement, or Doubling Theory, by focusing on the connections between the properties of the three-high and four-high towers, which is within the *property noticing* and *observing* level of understanding. Once they had completed building this arrangement, they immediately saw and justified the relationship between the different heights of towers, which took them into the *structuring* level of understanding.

Following Beth's explanation, I asked the girls how many five-high towers they could build. I assumed, based on Beth's justification of the relationship between the three-high and four-high towers, that the girls would state the answer, thirty-two, quickly. This was not the case, as it took them several minutes to come to that conclusion. Beth reasoned that "it could either add another one or double," and Ashley guessed that it would "add four new ones for sure," but neither girl could elaborate on their reasoning. A minute passed and Beth started to use the towers to guide her explanation:

Beth: Because for this one [pointed to BRRB], we could add, hold on. I gotta try this first [moved BRRB off to the side and then built RBRRB]. And then we could, for just this one [BRRB], we could add a red [placed RBRRB to the left of BRRB] or a blue on the other side [took one blue block and moved it toward the right side of BRRB (Fig. 13)]. So for each one of these [four-high towers], there's two more possible

Stacie: Okay, so how many were four-high?

All: Sixteen

Stacie: And then Beth, you said for each of these [four-high towers], what do you do?

Beth: Each of the sixteen there's two possibilities, red or a blue. [you can place either a red on top or a blue on top]. So then, it becomes thirty-two.

Jamie: So it just doubles each time pretty much?

Ashley: Yeah, it would like double.

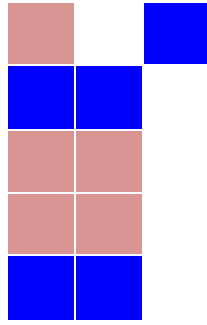


Figure 13: Beth's use of the towers to show the doubling.

In this exchange, Beth referenced back to the doubling arrangement they had built in order to explain that for each tower they had in a particular level, they could create two more “possibilities” or towers in the next height because they could add either a blue or red block onto each tower to create two higher towers. At this point she was still using the towers, the manipulatives themselves, to help her make sense of what this recursive pattern was, and that the number of towers would double with each subsequent height. She had to reference back to the way they had arranged their towers and check her reasoning with the five-high towers. Her use of the manipulatives helped her to explain and make sense of the recursive pattern. In this clip, Beth was working in the *property noticing*, *observing*, *structuring* and *inventising* levels of understanding. She focused on the properties of the towers and the properties of their arrangement, or Doubling Theory, in order to justify and explain the patterns she was seeing. She developed an inductive argument as to how the tower heights were related by elaborating on the properties she was seeing in her arrangement of the towers. After justifying their Doubling Theory the girls had proved how the number of towers of particular heights are related to each other, they had developed a solid understanding of this particular part of the task and were ready to move on, thus achieving the *inventising* level of understanding.

**Developing  $2^n$ .** After the girls had justified the Doubling Theory, I asked them to begin working on the second part of the task, which prompted them to determine how many  $n$ -high towers could be built. Beth immediately tried to apply the Doubling Theory they had developed. She reasoned that if they were to follow the doubling arrangement then the taller tower height would be  $n$ , so to get all of the towers of height  $n$  they would do  $n$  minus one, and then multiply that quantity by two. She explained an example using the connection between four-high and five-high towers.

Beth: So it would be like,  $n$  minus one [moves her finger from a five-high tower to a four-high tower], so we'd go down a level, and then times two to get back to all of the possibilities.

I asked Beth to write down what she was trying to say. She explained again that  $n$  was the height of the tower, so if they wanted all of the possibilities of that height, they would have to do  $(n - 1)$  to get down to a previous height and then multiply that quantity by two because they could add either a red block or blue block on top of each tower that was  $(n - 1)$  blocks high. Beth wrote it out in terms of  $n$ , which produced the following algebraic expression: Poss of  $N = (n - 1) \times 2$ . Beth applied the reasoning they went through while developing their Doubling Theory to produce this algebraic expression to find all of the towers that were  $n$  blocks high.

Beth was working in the *formalizing* level of understanding as she generalized they could assign a particular height of towers as their  $n$  height, and that they could find all of the towers of that height by taking  $(n - 1)$  to get to the previous height and then multiplying that quantity by two. Beth described the doubling relationship in terms of  $n$  and reasoned through notation that attempted to model that relationship. Although the notation Beth described was not accurate, Beth's abstract reasoning to describe the relationship was accurate and based on their use of the manipulatives.

As Beth went through and more formally explained her reasoning and developed an algebraic expression that represented her reasoning, she began to think of the notation as a theory and she moved from the *formalizing* to *observing* level of understanding. Up to this point in her reasoning, Beth was accurate in explaining that you have to multiply the number of towers of the previous height by two in order to get the number of towers in the subsequent height, but she had a hard time trying to figure out how to write it algebraically. In this exchange, Beth used the towers to help her abstract the algebraic nature of the relationship between the towers, but she had not abstracted enough to determine the correct algebraic notation. She had developed an algebraic theory based on the properties of the Doubling Theory and at the end of this exchange she successfully justified her theory, although algebraically, it was inaccurate at this point in the interview.

After Beth had written down the expression, the girls quickly realized that it would not provide the correct counts of towers that were  $n$ -high. Jamie suggested they should change the  $(n - 1)$  in the expression to  $(n + 1)$  because they were adding blocks onto any particular height to get to the next height, not removing blocks. They tested this reasoning and saw that it only worked for the case where  $n = 3$  but not for all cases of  $n$ . Ashley suggested it might be  $n^2$  but found that did not work either. After several minutes of reasoning through these different suggestions, Beth created a chart (Fig. 14) that accounted for the number of towers of a specific height  $n$  (note that the left column in her chart stands for the height of tower i.e. 3 stands for 3-high towers).

3-8  
 4-16  
 5-32  
 6-64  
 7-128

Figure 14: Beth's tower chart.

After writing down and analyzing this chart, Beth saw where they went wrong in their original algebraic expression, Poss of  $N = (n - 1) \times 2$ . Again, using the five-high towers as her  $n$  she reasoned:

Beth: So what if you take the possibilities of  $n$  minus one [writes " $(\text{Poss } n - 1)$ "], so all of them, not just... Instead of taking four, you'd take sixteen, and then you minus one. Times two [" $(\text{Poss } n - 1) \times 2$ "], that's thirty. . . unless you don't minus one. Yeah that would work. Say you're trying to figure out six high [writes "6 high"]. You take the possibilities of six minus one which is five, [points to 5-32] thirty-two. Those are all of the possibilities, times by two [writes " $(32) \times 2$ "], is sixty-four.

Ashley asked Beth to clarify what she had just said. Beth explained why the  $(n - 1)$  in their original notation of " $(n - 1) \times 2$ " was incorrect. Using the example of going from four-high to five-high towers, she said that  $(n - 1) \times 2$  would only account for using four of the four-high towers to create five-high towers. So, "instead of just those four [different four-high towers] having two different combinations, then all sixteen would have two [you would double all of the four-high towers, not just four of them]." Beth understood that they used a lower height  $(n - 1)$  in order to get towers of height  $n$ , but she did not know how to algebraically express all of the towers of height  $(n - 1)$ , which is what hampered the original expression she developed. After focusing on the relationship between their Doubling Theory and the counts of towers within a particular height, Beth adjusted her expression to  $(\text{Poss of } n - 1) \times 2$  so that she accounted for the doubling of each tower within the lower height of  $(n - 1)$ , not just  $(n - 1)$  towers within the lower height.

Here, Beth reasoned through her previous theory and developed a new generalization which corrected the algebraic expression she had originally developed to find all the  $n$ -high towers. Again, she was trying to abstract and find a way to algebraically express the connections between the heights of the towers. Her algebraic notation corresponded more closely to her prior reasoning that one would take the number of towers of a lower height and multiply that number by two to find all of the  $n$ -high towers. She developed a new and more accurate theory based on the properties of the Doubling Theory they had developed and justified her theory and notation by using specific examples of the towers and then generalizing it further without the towers. Beth was working in the *property noticing*, *observing* and *structuring* levels of understanding as she developed and justified of the correct algebraic notation based on the properties of their Doubling Theory.

After Beth explained the new expression “Poss of  $N = (\text{Poss of } n - 1) \times 2$ ” to Jamie and Ashley, she showed that the notation could accurately get them all of the towers that were three-high and seven-high tall. Because Beth had been using her chart of tower counts to help verify that the expression provided the correct number of towers, Ashley asked what they would do if they did not already know the number of towers within each particular height. The girls focused on the fact that the number of towers doubled from one height to the next, and that the doubling could be represented by repeated multiplication by two. Beth wrote out “ $(1 \cdot 2)2 \cdot 2 \cdot 2 \cdot 2$ ” which represented finding all of the six-high towers. She then shortened this notation to  $2^6$  and explained that the six represented towers that were six blocks high. Finally, Beth generalized this notation by writing “ $2^n = \# \text{ poss}$ ” and explained that  $2^n$  would give the “number of possibilities of everything.”

With Ashley's prompting, Beth shortened their expression "Poss of  $N = (\text{Poss of } n - 1) \times 2^n$ " into the expression  $2^n$ . While doing this Beth continued to work in the *observing* level of understanding as she attempted to synthesize their expression into one they could use without having to previously know the number of towers of any given height. She developed this expression  $2^n$  based off of the doubling property they had developed within their Doubling Theory. The expression was directly related to their use of the towers in building theories throughout the entire first interview. At the very end, Beth justified that  $2^n$  would provide them with all of the  $n$ -high towers, which moved her into the *structuring* level of understanding. The interview ended shortly after. The girls had progressed through the task and had developed an understanding not only of the original part of the task, finding all of the four-high towers, but they developed their Doubling Theory which helped them to create the expression to find all of the  $n$ -high towers. This culmination of the task resulted in their movement to the *inventising* level of understanding.

### **Manipulatives and Understanding in the First Interview**

**Students' use of the manipulatives.** Throughout the first interview, there were many ways in which the students used the manipulatives, or towers, to assist them in building understanding of the Towers Task. Some researchers have suggested that ability to manipulate and rearrange manipulatives can benefit students in making sense of mathematics (Hinzman, 1997; Resnick, 1998), but they did not provide explicit examples. Similar to these researchers, I found that the arrangements of the towers were important, but also that the attention the girls gave to specific properties of the towers is what prompted their ability to place towers into particular arrangements. Focusing on the properties and connections among the towers led the girls to build specific arrangements of towers. The particular arrangements of the manipulatives

were significant in helping the girls to develop the Cases, Doubling and  $2^n$  Theories. The girls justified they had all sixteen of the four-high towers only after having arranged the towers into cases. The Doubling Theory was a direct result of the girls arranging the towers in a specific way (Fig. 11 and 12) that helped them to see that they added a red or a blue block on top of each tower to get towers of the next height.

Research has suggested that manipulatives can help students move from concrete to more abstract reasoning (Chao et al., 2000; Chappell & Strutchens, 2001; Fusion & Briars, 1990). Similar to prior research, the girls' ability to abstract the properties from the doubling arrangements to develop the algebraic expressions to represent the  $n$ -high towers was an important part of their understanding. They developed the expression,  $2^n$ , by specifically relating back to the properties within the doubling arrangement they had created. The girls developed their theories and ideas as they explained the important aspects of each arrangement and then abstracted those ideas from the arrangements to build understanding of the Towers Task.

**Mathematics elicited by the manipulatives.** There were several important mathematical ideas that the girls developed through their use of the manipulatives during the Towers Task. The first piece of mathematics came out when the girls developed their Doubling Theory. Because of the particular arrangement they placed the towers in (Fig. 11 and 12), the girls created an inductive argument as to how the tower heights were related and how they could build all of the towers of a particular height. This inductive argument is similar to the inductive argument developed by the students in Maher and Martino (1996a), and was grounded on the particular arrangement of the manipulatives the girls used (Fig. 11 and 12). The girls were not able to develop a logical argument as to how the towers heights were related before placing the towers into that arrangement and noticing the properties within the arrangement.



A second piece of mathematics, which was similar to their inductive argument, was Beth's development of the idea of "possibilities." When the girls first developed their inductive argument, they based it on the fact that for each tower they could add a red or blue block on top to create the towers within the next height. This inductive argument was based on their ability to physically build the towers by placing a red or blue block on top of the lower towers. Beth extended this idea when she described how each tower of a lower height would produce two "possibilities" or two new towers of the next height. Beth's explanation of the different possibilities that a tower could produce was more abstracted than their induction idea of physically adding a red or a blue on top of each tower. Beth described her idea, of each tower creating two possibilities, without having to physically build the towers to show as an example and without even having to reference adding blocks on top of each tower.

A third piece of mathematics that was elicited was the girls' ability to algebraically express  $n$ -high towers. The girls developed three different algebraic expressions to try and represent these towers. The first of their expressions, although inaccurate, was based upon the Doubling Theory developed through manipulation of the towers. While some students may have stopped there and given up on the expression when they realized it was incorrect, in this interview, the girls moved forward and corrected their expression because they relied on the theories they had developed about the manipulatives to derive their expressions. Beth saw and explained the error in the quantity  $(n - 1)$  and corrected the error by replacing  $(n - 1)$  with  $(\text{Poss of } n - 1)$ . After justifying that the revised expression was accurate, the girls developed a more efficient form of that expression,  $2^n$ , by focusing on what the doubling meant in their theories and representing that doubling by repeated multiplication by two.

**Levels of Understanding.** Throughout the episodes in the first interview, the girls spent the majority of their time within the *property noticing*, *formalizing*, *observing* and *structuring* levels of understanding. Much of the task involved the girls seeing different patterns and properties within the towers or within the arrangements of the towers that they created. After seeing these properties, the girls then developed several different generalizations throughout the task, which led them into the *formalizing* level of understanding. Because the generalizations were based on the properties of the towers or arrangements of the towers, often times the girls continued *property noticing* while they were within the *formalizing* level of understanding. After making generalizations, such as arranging the four-high towers by case, the girls extended their generalizations and built theories based off of the generalizations, such as arranging the towers within each particular case in order to prove they had all of the four-high towers. This move toward thinking about their generalizations and ideas as theories led them into the *observing* level of understanding.

Each of the theories developed in the first interview were grounded on the work the girls were doing with the towers and the particular arrangements of the towers they had created. Even their development of algebraic expressions to describe the patterns in the task, such as  $2^n$ , was based on the towers and prior theories formed from their use of the towers. Also, the girls explained and justified their theories by elaborating on the properties they noticed within each theory, so oftentimes the girls participated in *property noticing* while they were within the *observing* level of understanding. Finally, with each particular theory that was developed, the girls justified different aspects of the theory in order to prove the theory accurate or not. Some of their theories were not accurate (such as the first algebraic notation) or were not useful to them

(the pyramid pattern), but nonetheless, the girls explained and justified their theories throughout the task, which moved them into the *structuring* level of understanding.

### Applying Towers to Pascal's Triangle

**Row Theory.** At the beginning of the second interview, I asked the girls to recall the work they had done with the towers task during the previous interview. They spent the first portion of the interview rebuilding all of the three-high and four-high towers, explaining their Cases Theory, Doubling Theory as well as their  $2^n$  Theory. After reviewing what they had done during the previous interview, I prompted the girls to arrange their four-high towers into an arrangement based on their Cases Theory. The girls placed the towers into the particular arrangement in Figure 15 by grouping the towers by case, according to the number of blue blocks contained in each tower. I then asked the girls to take out a piece of paper and write down how many towers had four blue blocks, three blue blocks, two blue blocks, one blue block and zero blue blocks. A recreation of the girls' chart is below in Figure 16.

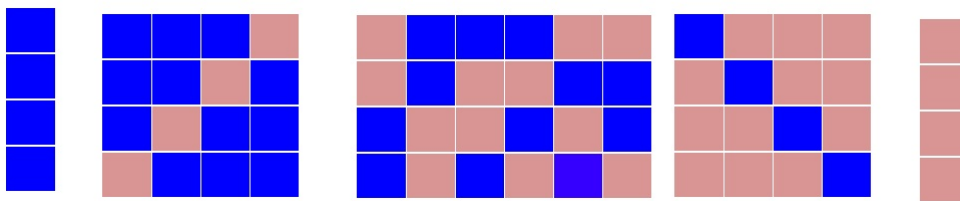


Figure 15: Arrangement of the four-high towers according to the Cases Theory

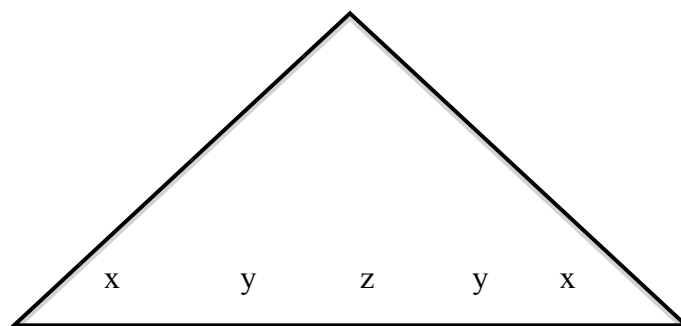
- 4 blue – 1 tower
- 3 blue – 4 towers
- 2 blue – 6 towers
- 1 blue – 4 towers
- 0 blue – 1 tower

Figure 16: Recreation of the girls' four-high tower charts.

After writing down their charts for the four-high towers, the girls produced a similar chart to account for all of the three-high towers that had 3 blue blocks, 2 blue blocks, 1 blue block and zero blue blocks (Fig. 17). After writing down the charts for the three-high and four-high towers, I asked the girls if they recognized the numbers they had recorded in the “towers” column of each chart. Ashley immediately made the connection between the towers and Pascal’s Triangle when she asked, “Wasn’t it that triangle thing?” I asked her to elaborate and she proceeded to draw an abstract representation of Pascal’s Triangle by forming an isosceles triangle on her paper and filling in the entries of a row using variables (Fig. 18).

3 blue – 1 tower  
 2 blue – 3 towers  
 1 blue – 3 towers  
 0 blue – 1 tower

*Figure 17:* Recreation of the girls’ three-high tower charts.



*Figure 18:* Recreation of Ashley’s initial representation of Pascal’s Triangle.

After creating her representation of Pascal’s Triangle (Fig. 18), Ashley pointed out a generalization she noticed about the symmetry within the triangle when she reasoned that the “last [entries] on each side were the same. . .” and that the pattern just “goes on and on and on.” Ashley was she was not exactly sure if her reasoning was correct. I asked her if she thought that

all of the entries inside the triangle would be variables, to which she responded while adding to her triangle representation (Fig. 19):

Ashley: No, no, yes, no, I don't know like. . . these [the x's in the triangle] could be like six, or let's use this example. One [writes "1" underneath the left x], one [writes "1" underneath the right x], four [writes "4" underneath the left y], four [writes "4" underneath the right y], six [writes "6" underneath the z] Or like this one [the numbers they got from the three-high towers], it would be up higher on the triangle [draws a cross section above x, y, z, y, x on her triangle], and it would be one, one, three, three maybe [writes 1 3 3 1 to the right outside of the triangle]. Or something like that.

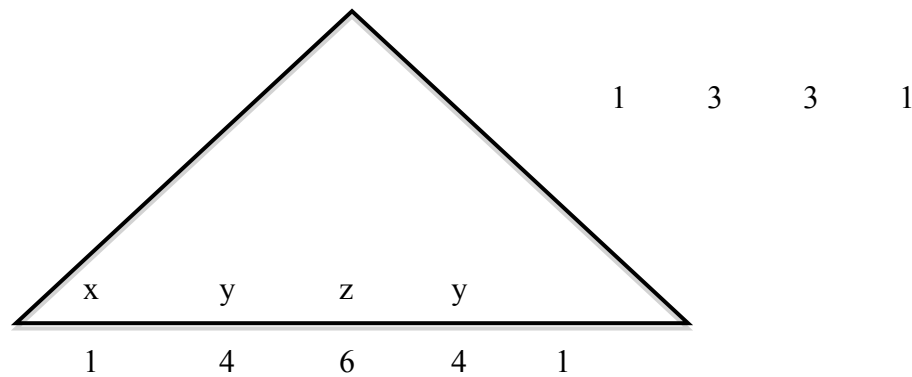


Figure 19: Ashley's extension of Pascal's Triangle.

In the beginning of the second interview, the girls were working within the *observing* level of understanding when they created the four-high towers and arranged the towers according to the Cases Theory, as well as when they abstracted properties about the Cases Theory to create numerical charts to represent the number of towers in each case. After being prompted as to whether or not they recognized the numbers within their charts, the girls folded back to the *prior knowledge* level when Ashley recalled that the numbers had to do with “that triangle thing.” When asked to explain her thinking in more detail, Ashley moved into the *formalizing* level when she made the generalization that the entries in Pascal's Triangle could be represented by variables, and that there was symmetry within the entries of the triangle. Ashley extended this

generalization by showing the connection between the Cases Theory and the tower counts in her chart with the variables she had written in the triangle. Here, Ashley developed the Row Theory, that the counts of towers within their charts could represent the rows in Pascal’s Triangle, which led to her movement into the *observing* level of understanding. The girls abstracted properties from their towers to create the numerical charts. They then made connections between the tower charts and Pascal’s Triangle, which helped them to theorize how the rows of Pascal’s Triangle could be represented by a particular arrangement of the towers.

After Ashley explained the Row Theory, Beth started to write down her own triangle. She created the four different two-high towers and found that there was 1 tower with 2 blue blocks, 2 towers with 1 blue block and 1 tower with 0 blue blocks. She recorded this information in a chart (Fig. 20, notice she has dropped the words “blue” and “towers”), then placed the additional information into the triangle she created (Fig. 21, notice the column on the far left represents the height of the tower in that particular row).

2 – 1  
1 – 2  
1 – 1

*Figure 20:* Beth’s shorthand chart for the two-high towers.

1)  
2) 1 2 1  
3) 1 3 3 1  
4) 1 4 6 4 1

*Figure 21:* Beth’s representation of Pascal’s Triangle.

**Fifth Row of Pascal’s Triangle.** Once she successfully filled out the second row<sup>1</sup> of Pascal’s Triangle represented by the two-high towers, Beth focused her attention on trying to fill out the fifth row. She reasoned that on either side of the row there would be a “one, five, then five, one [wrote out on her paper, 5) 1 5      5 1],” but she did not know what numbers would go in the middle of the row. Beth decided to create a chart similar to the chart she created for the two-high towers (Fig. 20) in order to find the other numbers that would go in the fifth row of the triangle. Beth explained there would be one tower with five blue blocks, five towers with four blue blocks, five towers with one blue block and one tower with zero blue blocks. Ashley then reasoned that total number of towers would add up to 32 because “two to the fifth is thirty-two.” Beth created the following chart (Fig. 22) and the girls realized the number of towers with two blue blocks and three blue blocks would have to add up to 20, but they were not sure whether the twenty towers would be split evenly between the two different groups of towers.

5 – 1  
 4 – 5  
 3 –  
 2 –  
 1 – 5  
 0 – 1

*Figure 22:* Beth’s shorthand chart for the five-high towers.

In this exchange, the girls continued to work in the *property noticing* and *observing* levels of understanding as they elaborated on their Row Theory, that the rows of Pascal’s Triangle could be represented by arranging the towers of each height by case, and recording how

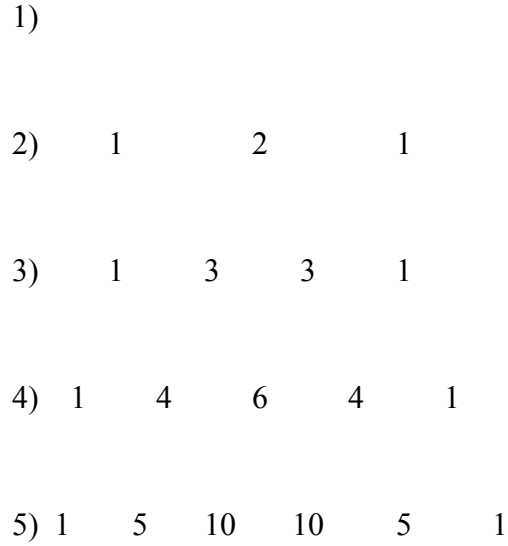
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<sup>1</sup> Throughout the analysis I will call this row the “second row,” as well as subsequent rows in a similar because it relates to how the girls were representing the rows of Pascal’s Triangle with towers of the corresponding height.

many towers of each case there were. The girls continued to make connections between their Cases Theory as to how to arrange the towers, and how the Cases Theory applied to particular rows of Pascal's Triangle. Beth filled in the second row of Pascal's Triangle by building the two-high towers and recording the properties she saw about the count of the towers after they were arranged by case. Then, without building the five-high towers, Beth reasoned about how many towers would be in each case as she started to fill in the entries for the fifth row of Pascal's Triangle. The arrangement of the towers according to the Cases Theory, and the properties they used from the theory to create the numerical charts helped the girls give meaning to the abstract rows of Pascal's Triangle. All of the numbers in each row of the triangle were built from their work with the towers, and the girls gave meaning to those numbers based on their Cases Theory.

Since the girls did not know how many five-high towers had two blue blocks and three blue blocks, they decided to build all of the five-high towers in order to complete their chart and fill in the corresponding row of Pascal's Triangle. The girls implemented their Doubling Theory while building the five-high towers. They made two different copies of each four-high tower and then added a blue block on top of one copy and a red block on top of the other copy. After building all 32 five-high towers, the girls arranged the towers into groups according to the number of blue blocks in each tower. They found there were 10 towers that had three blue blocks and 10 towers that had 2 blue blocks. Ashley filled out the rest of the chart and Beth completed filling out the fifth row in her representation of Pascal's Triangle (Fig. 23).





*Figure 23: Pascal's Triangle with the 5<sup>th</sup> row completely filled out.*

After Beth filled in the rest of the 5th row in Pascal's Triangle, the girls made a connection between the rows in Pascal's Triangle to their  $2^n$  Theory of how many towers they could build of each particular height. Beth wrote " $= 32$ " to the right of the fifth row in Pascal's Triangle and explained that it represented how many total five-high towers they could build. Beth then placed similar tower counts for the second, third and fourth row as well as filled out the 1<sup>st</sup> row of the triangle and accounted for the number of towers that were one block high (Fig. 24). When prompted to think about the towers that would be "zero" tall, the girls determined that in the very top row there would be a one by using the  $2^n$  Theory and knowing  $2^0 = 1$ . Beth explained that, "it would be one because you only have one option, nothing." She then placed a 1 in the row to account for the very top of Pascal's Triangle (Fig. 24).

0)				1						
1)			1		1			= 2		
2)		1		2		1		= 4		
3)		1		3		3		= 8		
4)	1		4		6		4	1	= 16	
5)	1		5		10		10	5	1	= 32

Figure 24: The completed version of Pascal's Triangle.

Here, the girls continued to work in the *property noticing*, and *observing* levels of understanding, but they also moved into the *structuring* level of understanding when they justified what the sum of each row in Pascal's Triangle would be and that the top row would be one. Throughout this span of the interview, the girls implemented several different theories in order to fill out the fifth and top row of Pascal's Triangle. First, they used the Doubling Theory in order to create all of the five-high towers. Then they arranged all of the five-high towers according to their Cases Theory in order to determine how many towers had 3 blue blocks and 2 blue blocks. After building and arranging the five-high towers, the girls then completed their fifth row of Pascal's Triangle, applying their Row Theory. The girls also applied their  $2^n$  Theory in order to determine what each row of the triangle would sum to, as well as what the top row was. While using all of these theories, the girls were within the *observing* level of understanding. It is interesting to note, that while they were in the *observing* level in this instance, they were

making connections between the properties of several different theories and not just one particular theory, which was the case in much of the first interview. Here, the girls were abstracting properties from the Doubling Theory, Cases Theory and Row Theory in order to make sense of what the rows in the triangle meant and to create the 5<sup>th</sup> row. Then the girls related the numbers within each row of the triangle back to the  $2^n$  Theory to justify the sum of each row and what the top row would look like. These justifications moved the girls into the *structuring* level of understanding.

**Traditional Addition Rule.** After the girls had finished filling in the first five rows of Pascal's Triangle, I asked them how they might find the rest of the rows of the triangle. The girls focused on the numerical entries of each row to try and determine a pattern within the numbers. Ashley tried to recall how they found the entries of each row using a basic operation and she reasoned they could use multiplication somehow, but could not elaborate on this reasoning. After several minutes, Beth recognized the pattern of the standard addition rule and started to fill out the sixth row of Pascal's Triangle (Fig. 25). Beth explained how she recognized the pattern starting in the third row of the Triangle:

Beth: Because, three plus three is six, four plus six is ten, so then if we keep going [pointing at ten and five], that would be fifteen.

$$\begin{array}{r}
 \cdot \\
 \cdot \\
 \cdot \\
 3) \quad 1 \quad 3 \quad 3 \quad 1 \quad = 8 \\
 4) \quad 1 \quad 4 \quad 6 \quad 4 \quad 1 \quad = 16 \\
 5) \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \quad = 32 \\
 6) \quad 1 \quad 6 \quad 15 \quad 15 \quad 6 \quad 1 \quad = 64
 \end{array}$$

Figure 25: A portion of Pascal’s Triangle, including the partially filled out 6<sup>th</sup> row.

The girls were confused as to what number would go between the two fifteens. They recognized that the numerical entries of the row would have to add up to be 64 because  $2^6 = 64$  so they reasoned that they were missing twenty towers. Beth asked whether the 20 would be split between two entries or was just one entry. Ashley reasoned that it should not be split because there were six entries in the fifth row so there should be seven entries in the sixth row. Beth placed the number “20” between the fifteens in the sixth row (Fig. 26) and reasoned that it couldn’t be split “because this one [20] has to be bigger than this [15].” I asked her why that would be the case and she replied that it just had to be more.

$$6) \quad 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \quad = 64$$

Figure 26: The 6<sup>th</sup> row of Pascal’s Triangle.

When Beth did not explain any further, I asked her what the leftmost one in the sixth row meant in terms of the towers. She responded that it represented one tower that had all blue blocks. Beth then went on to explain that the left most six meant that there were six towers with five blue blocks and that number of blue blocks would decrease by one with each number in the row until she got to the right most one and explained that it represented the one tower with no blue blocks. This helped them to realize they had accounted for all of the groups of towers in the sixth row and that the 20 would not be split between two entries. After filling out the sixth row, the girls used the traditional addition rule to create the seventh row of Pascal's Triangle.

Throughout the last few minutes of the second interview the girls worked within the *property noticing, formalizing* and *observing* levels of understanding. When asked how to find other entries of Pascal's Triangle, the girls had to fold back from the *observing* level to the *formalizing* as they could not quickly find the other entries by applying theories they had already developed. The girls reasoned about the patterns they were seeing in the numbers of Pascal's Triangle, rather than trying to extend or apply prior theories, to find the entries of the triangle. Beth recognized the pattern of the traditional addition rule and generalized that it may be able to help them find the other entries. Beth explained the addition rule using specific entries within the triangle. Throughout the exchange, Beth was focusing on the properties she noticed in the numerical patterns within Pascal's Triangle. When the girls got stumped when filling out the sixth row of the triangle, they related the Cases Theory of the towers in order to help them accurately fill out the sixth row. The girls moved into the *observing* level of understanding when they made the connection between their Cases Theory and the entries in the sixth row of the triangle in order to justify they had correctly filled out that row. Here, the girls developed the generalization of the addition rule based solely on the number patterns they saw within their

numerical representation of Pascal's Triangle. When the girls had confusion as to how to many entries would be in the sixth row of the triangle, they related what that row of the triangle would mean in terms of the towers. Making the connection between the sixth row of Pascal's Triangle and how it was related to the Cases Theory of the towers helped move the girls into the *observing* level of understanding as they used the properties of the Cases Theory in order to help them know how many entries would be in that row.

It should be noted that although the girls were able explain the traditional addition rule at the end of this second interview, their explanation was based solely on the pattern they saw within the numerical entries between the third, fourth and fifth rows of Pascal's Triangle. The girls did not spend any time thinking about why the addition rule works in this interview, which is the context of the third interview. As a reader, one might assume that during the third interview the girls would focus on justifying the addition rule in terms of adding together adjacent numbers to find the entries in subsequent rows of the triangle, but this does not end up being the case instead, the girls develop their own unique conceptions of the addition rule in terms of the towers.

### **Manipulatives and Understanding in the Second Interview**

**Students' use of the manipulatives.** Throughout the second interview, there were several different ways in which the students used the towers to assist them in developing understanding of Pascal's Triangle. One significant use of the manipulatives was to use the arrangement of the towers, which builds on the research that the manipulability of manipulatives can enhance student learning (Clements, 2000; Hinzman, 1997; Resnick, 1998). The particular arrangement of the towers, according to the Cases Theory, helped the girls create numerical charts based on the number of towers within each case. These charts were then directly applied

toward creating the rows of Pascal's Triangle. Other important uses of the manipulatives were focusing on the properties of the Cases Theory and abstracting those properties in order to make connections between the towers and the rows of Pascal's Triangle, which corresponds to Moyer's (2001) argument that the physical nature of manipulatives do not carry the meaning of the mathematics and that students must reflect upon their actions with the manipulatives to create meaning. Although the physical placement of the towers according to the Cases Theory was important, the girls had to abstract properties of the Cases Theory in order to create the numerical charts based on each case. The girls then made the connection that the rows of Pascal's Triangle could be represented by the cases of each height of the towers, which helped the girls to give meaning to an abstract concept such as Pascal's Triangle. Finally, throughout the interview, the girls used the towers and Cases Theory to reason about and explain what the entries of Pascal's Triangle meant in terms of the towers.

**Mathematics elicited by the manipulatives.** The important mathematics elicited in the second interview was related to the girls' ability to give meaning to abstract expressions created from their use of the towers and the Cases Theory. The girls abstracted properties from the Cases Theory in order to create algebraic expressions in the numerical charts they created as well as in the numerical representation of Pascal's Triangle. Because the girls developed these expressions based on the towers, the expressions made sense to the girls and were easy for the girls to explain and justify in terms of the towers.

**Levels of Understanding.** Throughout the majority of the second interview the girls worked within the *property noticing* and *observing* levels of understanding. Although there were a few instances where the girls developed generalizations while working in the *formalizing* level or justified their theories to move to the *structuring* level, most of this interview was focused on

the girls making connections between different theories to develop an abstract expression for Pascal's Triangle and give meaning to the numerical entries of the triangle based on the towers.

As opposed to the first interview, most of the work the girls did the *property noticing* level was focused on analyzing the properties of different theories rather than analyzing the properties of specific towers or arrangements of towers. While working within the *observing* level of understanding, the girls made connections between several different theories. After the girls had used their Cases Theory to arrange the towers and record the numerical charts to account for the number of towers in each case, Ashley recognized the numbers in the chart and introduced the idea of Pascal's Triangle. This prompted the development of the Row Theory, that the towers, arranged by case, represented the rows of the triangle. The girls filled out the third and fourth rows of the triangle from their numerical charts and Beth elaborated on this symmetrical generalization to begin filling out the edges of the fifth row of the triangle, but had trouble determining what entries would go in the middle of that row. This prompted the girls to relate the triangle back to the towers, and specifically the Cases Theory. After building all of the five-high towers, the girls finished filling in the fifth row of the triangle based on the Cases Theory. Ashley incorporated the theory about  $2^n$  into this interview when she pointed out that the numbers within each row of the triangle would correspond to the number of towers of a particular height. The girls connected all of these theories together in order to help them create the sixth row of Pascal's Triangle.

### **Developing the Addition Rule of Pascal's Triangle**

At the start of the third interview, I asked the girls to review what the number entries in the rows of Pascal's Triangle meant. Beth immediately explained the Row Theory they developed, that the numbers within each row represented towers of a particular height with a



particular number of red and blue blocks. For example, she pointed out that the right three in the third row of the triangle represented three-high towers, specifically that there were three such towers that had exactly two blue blocks in them. After Beth explained what the number entries meant in terms of the towers, the girls recalled the addition rule they had discovered from analyzing the number patterns they saw in their numerical representation of Pascal's Triangle. I asked the girls to prove why the addition rule works.

The girls immediately tried to explain how the addition rule would apply to the towers. Ashley suggested that the addition of the numbers had to do with adding red and blue blocks on top of particular towers in order to create the towers that would go in the next row of the triangle. I asked the girls if they could extend Ashley's generalization. They spent several minutes trying to elaborate the generalization based on the relationship between the seven-high and eight-high towers that represented the seventh and eighth row of Pascal's Triangle respectively, but they were unable to make explain Ashley's generalization any further. Since the girls could not progress with analyzing the seven-high and eight-high towers, I suggested they could think about smaller cases such as two-high or three-high towers. Beth built off of Ashley's original generalization when she explained that you would add a red and a blue block on top of the three-high towers to get the four high-towers so by "adding those two amounts of towers [points to two adjacent numbers in Pascal's Triangle] together will get how many there are."

As the girls reviewed what the number entries of Pascal's Triangle meant, they were working in the *property noticing* and *formalizing* levels of understanding. They explained that each row of the triangle represented towers of a particular height, and that the numbers within each row represented the towers with a particular number of blue blocks in them. The girls were prompted to fold back when I asked them to prove why the addition rule works. Ashley

developed a generalization that the addition rule was related to the Doubling Theory of adding a red or a blue block on top of each tower to create subsequent heights of towers, which is evidence of *formalizing*. Although the girls were close to developing a theory as to how the addition rule related to how they built the towers, Beth's final explanation, of adding together adjacent numbers in the triangle, was not detailed enough to be considered a theory, thus they had not moved back into the *observing* level of understanding at this point.

**Physical Splitting Theory.** Since Beth could go no further with her generalization, I asked her where the RRR tower would be placed in Pascal's Triangle. Beth placed RRR on the left "1" in the second row of Pascal's Triangle<sup>2</sup>. I then asked her how she could create four-high towers from RRR. Beth explained that you would add either a red or a blue block on top. She then built RRRR and placed it on the left "1" in the third row of the triangle and then built BRRR and placed it on top of the left three in the third row. Ashley realized that Beth had incorrectly placed the RRR on the second row and that it should really be placed on the left "1" in the third row. Beth moved RRR down to the left "1" in the third row and also moved RRRR and BRRR down to the fourth row. Beth got excited about this new generalization and said, "I think this might work," before suggesting they try it with the two-high towers. Beth built RR and placed it on the left "1" in the second row. She then built RB and BR and placed them on the two and in the second row, as well as BB and placed it on the "1" in that row (See Fig. 27).

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<sup>2</sup> There was some confusion as to whether or not the RRR should be placed on the right "1" or left "1" of the second row. After a couple of minutes of discussion the girls decided to proceed by placing RRR on the left side of the triangle.

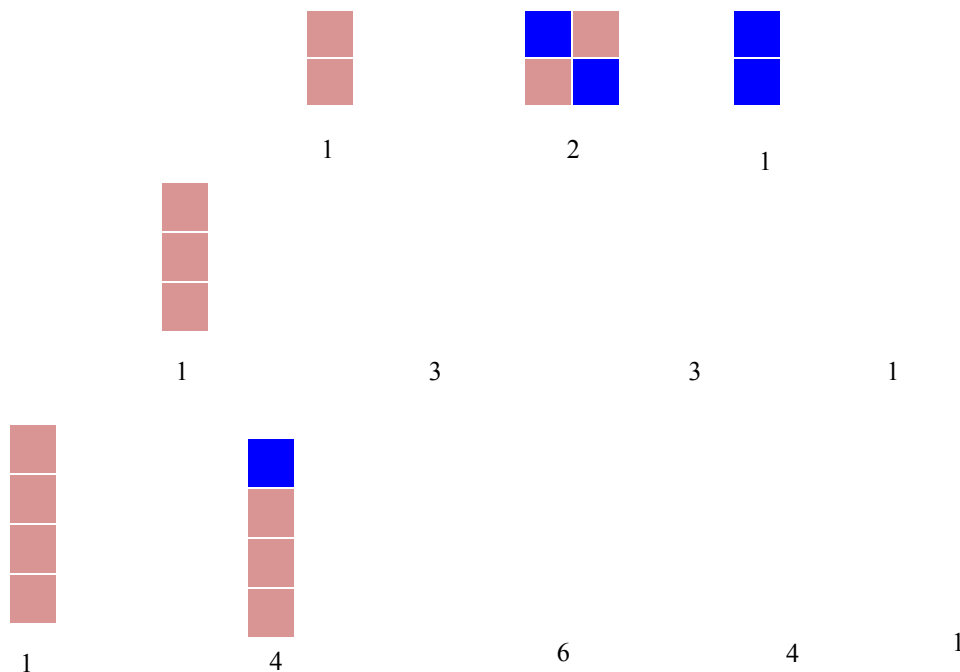


Figure 27: A recreation of how Beth placed towers directly onto Pascal's Triangle.

Ashley asked Beth to explain what she was doing. Beth explained the Physical Splitting Theory and also extended it to apply to the relationship between all of the two-high and three-high towers:

Beth: What I've done is . . . We have this tower [RR] and we can add a red to it, to make it no blue, or we could add a blue to it to put it in here [See Fig. 28]. So by adding . . . Oh, and add a red on top of that [BB] and then if I were to add a blue [onto BB], it would go there [See Fig. 28]

Beth then explained that you could create two three-high towers, RBR and BBR, from BR. She placed RBR with BRR on top of the left three in the third row and BBR with RBB on the right three. Beth then created RRB and BRB from RB and placed RRB on the left three and BRB on the right three (Fig. 29). After building and arranging all of the three high towers within Pascal's Triangle, Beth wrote down labels below the entries in the third row. She wrote "0 blue" underneath the left one, "1 blue" underneath the left three, "2 blue" underneath the right three

and “3 blue” underneath the right one. I asked her to label the second row in a similar way (Fig. 29).

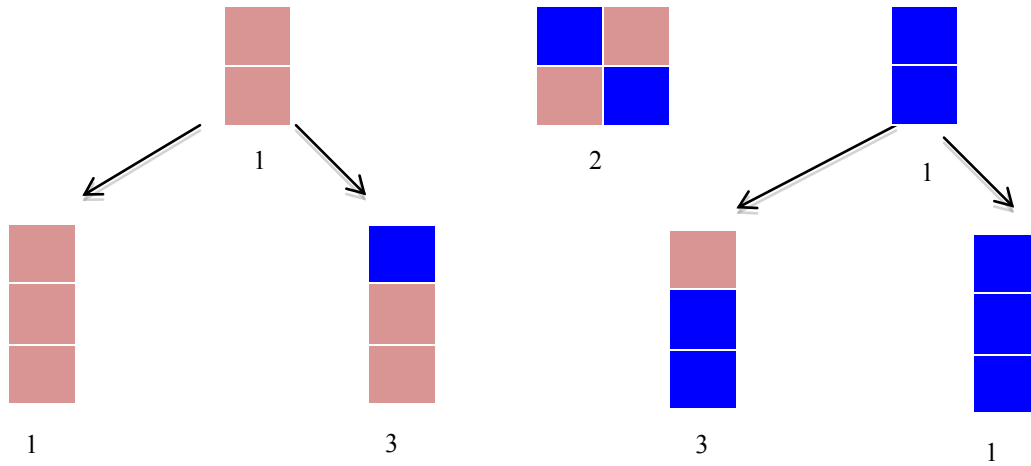


Figure 28: Recreation of how Beth placed towers directly onto Pascal's Triangle.

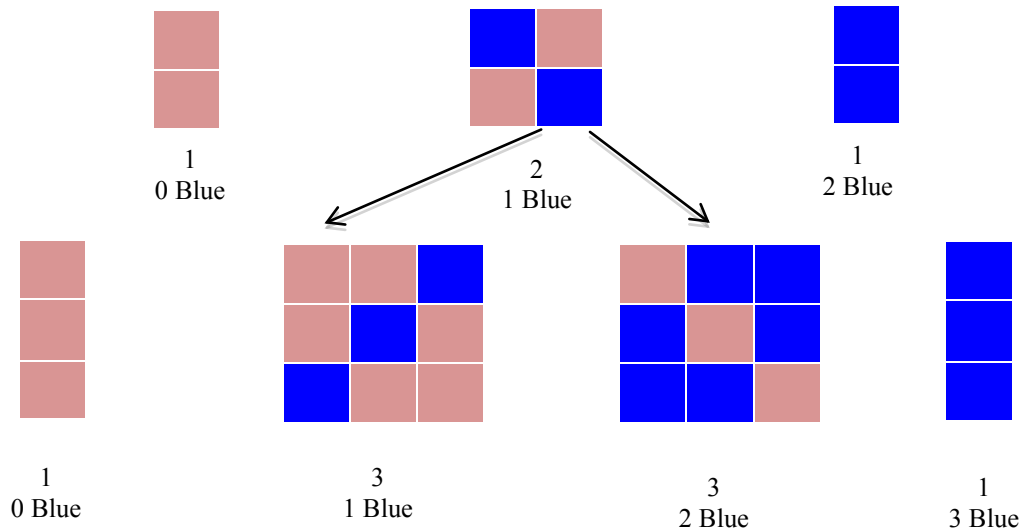


Figure 29: Recreation of the towers placed on the 3<sup>rd</sup> row and labels for the 2<sup>nd</sup> and 3<sup>rd</sup> rows.

Here, Beth moved into the *property noticing* and *formalizing* levels of understanding. She focused on the properties of the towers and where they would be placed in Pascal's Triangle based on the number of blue blocks in each tower. She created a new generalization that the towers could be physically placed on the numbers within Pascal's Triangle. Up to this point, the girls understood that the numbers in the triangle represented particular groups of towers, but they had not made a physical representation showing the direct connection between the towers and the numbers within each row of the triangle. Beth developed this generalization into the Physical Splitting Theory and moved into the *observing* level of understanding when she stated, "I think this might work. . ." and then explained how the Doubling Theory, applied to the two-high towers, would influence where the new three-high towers would go in the third row of Pascal's Triangle. Ashley asked Beth to explain the Physical Splitting Theory again, and Beth moved into the *structuring* level of understanding when she justified the theory using the specific examples of creating three-high towers from RR and BB and then showing how those three-high towers would be placed within the third row of Pascal's Triangle.

The towers were significant in helping the girls build understanding about the connections between the rows of Pascal's Triangle. After building the two-high towers and physically arranging them on top of the numbers in the second row of the triangle, Beth explained and justified the Physical Splitting Theory, that adding a red and a blue block on top of the two-high towers would influence where the resulting three-high towers would be placed in next row of the triangle. This theory Beth created was based upon the connections she made between the properties of two different theories they had developed about the towers during the first and second interviews respectively. The first theory was the Doubling Theory as she focused on the property of adding a red or a blue block on top of each two-high tower. The

second theory was the Row Theory which explained what each of the numbers in the rows of Pascal's Triangle meant in terms of the towers. By connecting these two theories Beth developed and justified the Physical Splitting Theory. Although most of the mathematics in this exchange is focused on what the numbers of Pascal's Triangle mean, the Physical Splitting Theory provided a foundation for the girls to justify why the addition rule works.

**Abstract Splitting Theory.** After Beth's explanation of the Physical Splitting Theory, Ashley got excited and said that she really liked it. I asked her to explain why she liked the theory and she tried to explain how it worked. She summarized that when you added a red or a blue block on top of a particular tower it influenced where the subsequent tower would be placed in Pascal's Triangle, but she could not elaborate any further. Beth extended Ashley's explanation and related the Physical Splitting Theory to why the addition rule works:

Beth: Okay I think I know what you mean. To get from here [BR] you could add a red or a blue [block on top], so by adding a red it goes in here and by adding a blue it goes in here [See Fig. 30]. And then here [RB] if you add a red it goes in here, if you add a blue it goes in here [See Fig. 31]. And since to make it from two [high towers] to three [high towers], it like multiplies by two . . . it just kind of splits up because it doubles, so half that would come from here [BB] go to each one [See Fig. 32].

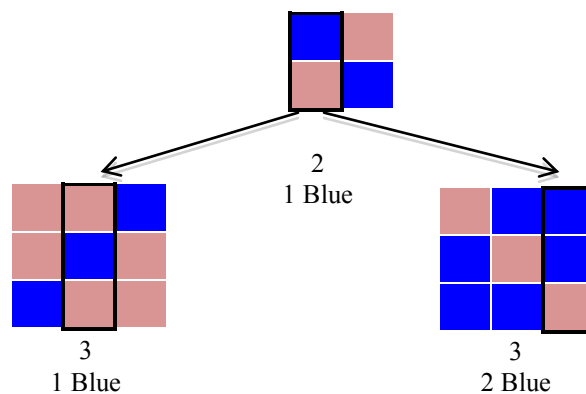


Figure 30: Beth's representation of the addition rule based on the towers.

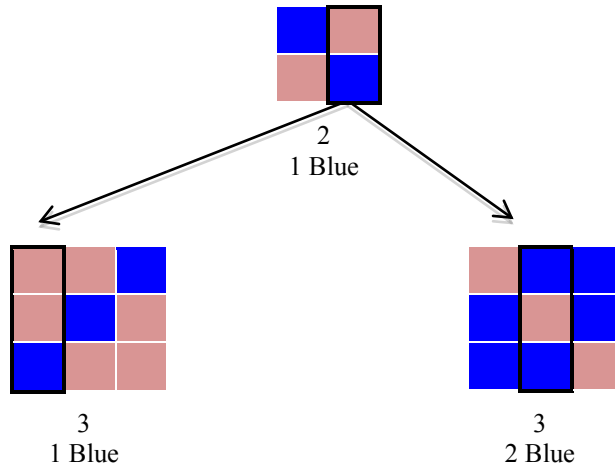


Figure 31: Beth's representation of the addition rule based on the towers.

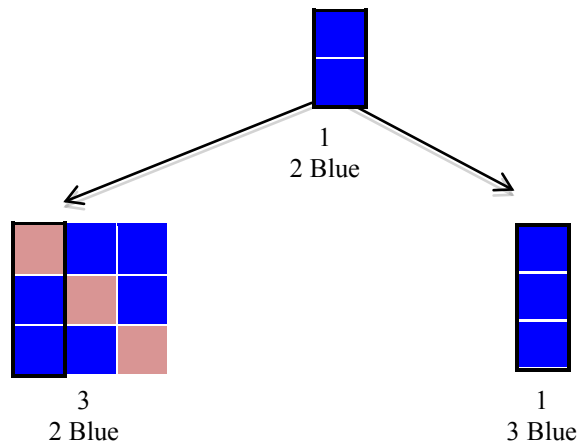


Figure 32: Beth's representation of the addition rule based on the towers.

I asked Beth to explain what she meant by “half” of the towers going to each group, which led her to extend the Physical Splitting Theory into an Abstract Splitting Theory. She used a specific case with BR and said that to make it three blocks high, you would create two more towers RBR and BBR. She then explained that one tower, RBR, which is half of the two new towers created, would go into the group of three towers with one blue block in the third row. She explained that the other half of the towers, BBR, would go in the group of three towers with

two blue blocks (Fig. 30). Beth used similar reasoning to explain that from RB, half of the three high towers would go in the group of three towers with one blue block, and half of the towers would go in the three towers with two blue blocks (Fig. 31). She justified that this halving, or splitting of the towers into the adjacent groups in the next row of Pascal's Triangle was what accounted for the addition rule because that "halving" would occur for all of the towers in every single row of the triangle.

After Beth's explanation of the Abstract Splitting Theory, the girls decided to build the four-high towers and place them onto Pascal's Triangle. They created the four high towers by following their Doubling Theory; they duplicated each of their three high towers and then added a red block and blue block onto each three-high tower. They placed the four high towers onto the fourth row of Pascal's Triangle according to how many blue blocks were in each tower. I then asked Jamie to explain how RBBB ended up in the group of six towers with three blue blocks. She explained that RBBB came from adding a red block on top of BBB, so it would be within that group. Ashley and Beth used similar explanations to justify how RRBB and RRBR ended up in the group of six towers with two blue blocks and four towers with one blue block.

Within this final exchange, the girls were working within the *property noticing*, *observing* and *structuring* levels of understanding as they explained and justified the the properties of three different theories. After being prompted by Ashley, Beth again justified the Physical Splitting Theory. Beth then extended and generalized this theory to form the Abstract Splitting Theory, in order to justify their conception of the addition rule. She made a reference back to their Doubling Theory when she explained that the number of towers "multiplies by two" or "doubles," and then she connected the Doubling Theory to the Abstract Splitting Theory, that half of the new towers created would go into the adjacent groups in the next row of the triangle.



Beth went on to justify the Abstract Splitting Theory using specific examples of creating the three-high towers from BR and RB and showing that the three-high towers get split into the adjacent groups in the third row of the triangle. After Beth made connections between, and justified these different theories, each of the girls explained how particular four-high towers ended up on top of specific numbers of Pascal's Triangle according to those theories they justified.

In this final exchange, the towers helped the girls to develop a justification as to why the addition rule works. Here, Beth explained and justified the connection between the Doubling Theory of building towers, to the properties of the arranging the towers on Pascal's Triangle by case. Although Beth originally used specific cases of towers to show how the Doubling Theory related to the Physical Splitting Theory, she abstracted the pattern she was seeing when she explained the Abstract Splitting Theory. Up to that point in the interview, everything was centered on physically showing the patterns with the towers using specific examples, and although Beth's explanation of the Abstract Splitting Theory was still based on the towers, it was an abstracted explanation that could have applied to towers within any row of Pascal's Triangle.

**Physical Splitting Theory vs. Abstract Splitting Theory.** The emergence of the Physical Splitting Theory and Abstract Splitting Theory occurred in the third interview. These two theories are similar in many ways, so it is important to point out the distinction between these two theories at this time. Both of the theories are based on the Doubling Theory, but upon different properties of the Doubling Theory. The Physical Splitting Theory is based on the physical aspect of the Doubling Theory, that a red block or a blue block could be added to towers of a specific height in order to form towers of the subsequent height. Beth explained the Physical Splitting Theory by using the example of RR. She said RR could be used to create towers in the

third row by taking RR and adding a red or a blue block on top of it. This would create RRR and BRR, which would then be placed in their respective groupings in the third row. The Physical Splitting Theory relates to physically building the towers according to the Doubling Theory, and then seeing how adding a red or blue block on top of a tower will influence where the new towers are placed in the next row of Pascal's Triangle.

While the Physical Splitting Theory was based on the physical properties of the Doubling Theory, the girls based the Abstract Splitting Theory on the abstract properties of the Doubling Theory, that they would multiply the number of towers of a particular height by two in order to find the number of towers of the next height. In explaining the Abstract Splitting Theory, Beth said half of the towers created from a particular tower in one row would be placed in one of the adjacent groups of the next row, and the other half of the towers would be placed in the other adjacent group. Not only was this theory formed from an abstract property of the Doubling Theory, but it was also a more abstract theory itself. The girls' verbal explanation of this theory was more abstract as compared to their explanation of the Physical Splitting Theory, which was grounded on physically building and manipulating towers.

### **Manipulatives and Understanding in the Third Interview**

**Students' use of the manipulatives.** Throughout the third interview, there were several different ways the girls used the manipulatives in order to facilitate their understanding of the addition rule. Building the towers and arranging the towers within Pascal's Triangle was important during the whole interview, but it was especially significant toward the beginning of the interview. For the first several minutes, the girls tried to justify why the addition rule works, but they struggled to develop any arguments. Once Beth realized she could build and arrange the towers by case, on top of Pascal's Triangle, the girls progressed further with their ideas and

developed theories about the addition rule based off the connections between rows of towers in the triangle. This is another explicit example of how the arrangement of the manipulatives was useful in helping the girls to continue moving through the levels of understanding.

Another important use of the manipulatives came from the girls explaining properties and making connections between different theories based on the towers. This use of the manipulatives was especially important when Beth made connections between the Doubling Theory (that the number of towers doubles from one height to the next) and Abstract Splitting Theory (dividing towers in “half” among adjacent groups) in order to explain their conception of the addition rule. Finally, at the end of the interview, the girls abstracted their theories in order to justify how the addition rule worked in terms of the towers. The progression of the abstraction in this interview is slightly different than how it is addressed in the literature. Some research has suggested that students should progress from using concrete materials to using abstract notation or symbolic representations (Ambrose, 2002; Bruner 1966; Piaget, 1952), but here the girls were using the manipulatives at the same time as they were abstracting their explanations. The girls’ abstraction was based on the physical representation of the towers in the rows of Pascal’s Triangle, but came from their verbal explanations of how the towers would “split” and “halve” into adjacent groups in the next row. They were working in a high level of understanding and using the manipulatives at the same time that they were abstracting.

**Mathematics elicited by the manipulatives.** The addition rule of Pascal’s Triangle was the most important mathematical idea that was developed in the third interview. It is interesting to point out here that the conceptions that the girls developed of the addition rule are unique to those of the traditional addition rule. At the end of the second interview, the girls described traditional addition rule after they recognized the patterns in the numbers of Pascal’s Triangle. In

the third interview, instead of focusing specifically on why one would add together adjacent numbers in the rows of Pascal's Triangle, the girls created their own conceptions of the addition rule based on the towers. To the girls, the addition rule meant taking every tower of a particular row and adding a red block or blue block on top of that tower, then depending on whether a red or blue block was added, placing half of the new towers in one adjacent group of the next row and half of the towers in the other adjacent group. Although the girls never made the connection to how many towers would get added to each group, or why you would add together adjacent numbers to form the new number in the next row, they still gave meaning to the addition rule based on the towers, which is consistent with their developing the rows of Pascal's Triangle from the towers during the second interview.

**Levels of Understanding.** During the third interview, the girls worked within several different levels of understanding. Toward the beginning of the interview the girls were within the *property noticing* and *formalizing* levels of understanding when Beth suggested they could physically place towers on top of the numbers within the rows of Pascal's Triangle according to what the numbers meant in terms of the towers. The girls then moved into the *observing* level of understanding when they extended this generalization to form the Physical Splitting Theory, that adding a red or blue block on top of each tower would influence where the new towers would be placed in the next row of the triangle. The rest of the time spent in the *observing* level of understanding came from the girls making connections between several different theories as they attempted to make sense of the addition rule. The girls moved into the *structuring* level of understanding when Beth justified the Abstract Splitting Theory and explained how it accounted for the addition rule in Pascal's Triangle.

In this chapter, I have provided a narrative of compelling episodes that illustrated how a group of students used manipulatives to facilitate their growth of understanding of the Towers Task and of Pascal's Triangle. The students developed understanding of these topics by creating theories, based from the properties of the towers and arrangements of towers, in order to justify their ideas and reasoning. The students' progression through different levels of understanding was prompted by their use of the manipulatives and will be discussed in more detail in the following chapter.

## Chapter 5: Conclusions

The purpose of this case study was to determine how manipulatives facilitated one group of high school students' understanding of combinatorics and Pascal's Triangle over time. These data and analysis suggest a few different themes in relation to the students' use of manipulatives to facilitate understanding. First, the girls' understanding progressed through several different levels of understanding within the Pirie-Kieren model, and the manipulatives facilitated this progression. Second, the girls' ability to make connections between theories allowed them to continue working in outer levels of understanding without necessarily having to fold back. Third, the manipulatives did not impede on the girls' ability to develop more sophisticated and abstract justifications throughout the interviews. The analysis also suggests some extensions to particular aspects of the Pirie-Kieren Theory for Growth of Mathematical Understanding.

### **Manipulatives Facilitated Growth of Understanding**

The literature on manipulatives has addressed how manipulatives may be useful to students. Some studies have shown that manipulatives help student to progress from concrete representations to more abstract representations (Chao, Stigler, & Woodward, 2000; Chappell & Strutchens, 2001; Ambrose, 2002). Others have claimed that manipulatives can be helpful for students as they can relate their own physical experiences to mathematics (Carraher, Carraher & Schliemann, 1985; Kamii, Lewis, & Kirkland, 2001). Some research suggests that manipulatives are useful as they engage students in mathematical activity (Moch, 2001; Moyer, 2001), and others argue that the manipulability of manipulatives promote student learning (Clements, 2000; Hinzman, 1997; Resnick, 1998). Although the literature has addressed how manipulatives may be useful to students, especially elementary students, it has failed to explicitly address how manipulatives can help high school students' build mathematical understanding.

The literature also lacks explicit examples of how manipulatives facilitate mathematical understanding, and within this case study, the role of the manipulatives becomes clearer for these high school students as they developed understanding of combinations and Pascal's Triangle. Within the first interview, the girls created theories that helped them to build understanding of how to represent the number of combinations of four distinct objects, taken two at a time and the number of combinations of  $n$  distinct objects taken two at a time. The girls also justified that the sum of the combinations of towers of a given height  $n$ , is  $2^n$ . Within the second interview, the girls made connections between Pascal's Triangle and the towers, which led them to develop understanding of what the numbers in each row of the triangle mean. After giving meaning to the rows of Pascal's Triangle, the girls formed their own unique understanding of the addition rule by forming new theories based on the properties of prior theories they had developed from the towers.

Throughout all of the interviews in this case study, there was a pattern to the progression of how the girls moved through the levels of understanding in the Pirie-Kieren model and this progression was a direct result of the girls' use of the manipulatives in the interviews. The girls would begin in the *property noticing* level of understanding where they described properties of the towers or particular arrangements of towers. By making connections between these properties, the girls developed generalizations about the towers or arrangements, which led them into the *formalizing* level of understanding. Once the girls developed generalizations, they often elaborated the generalizations by explaining different properties or patterns they saw within the generalizations. This led the girls to view the generalizations as theories and helped them progress into the *observing* level of understanding. Finally, once the girls had developed theories, they moved into the *structuring* level of understanding where they justified whether or not their

theories were accurate. Throughout all of the interviews, this progression of moving through the levels of understanding played an important role in helping the girls develop understanding of combinatorics and Pascal's Triangle, and the progression was facilitated by the girls' use of the manipulatives.

**First Interview.** During the first interview, this progression of noticing properties, forming generalizations and theories, and then justifying theories can be seen when the girls were developing their Cases Theory and Doubling Theory. The Cases Theory was a result of the girls attempting to prove they had all of the four-high towers. It began in the *property noticing* level of understanding when the girls noticed particular properties within the four-high towers and started to point out that which colored block was in each spot of the towers. They moved into the *formalizing* level of understanding when they generalized that the towers could be organized by grouping the towers together by case according to how many blue and red blocks were within each tower. Once the girls had arranged the towers by case, they noticed properties about this particular arrangement, such as a property that the red block could be placed within each spot in the 3B1R towers, and that in the 2B2R case, one blue block would be held constant in the bottom spot while the other blue block would move through each other spot in the tower. These explanations of the different properties of each case of towers are evidence that the girls were now viewing their cases arrangement as a theory and that they had moved into the *observing* level of understanding. The girls moved into the *structuring* level of understanding when they justified they had all of the towers of each case based on the properties they noticed about the towers and the ways the towers were arranged in each case. The girls proved they had all sixteen of the four-high towers by justifying they had all of the towers within each case.



The girls went through a similar progression of understanding as they tried to justify the how they could create four-high towers from the three-high towers. Before creating the Doubling Theory that helped them to justify this relationship between the towers, the girls developed a few other generalizations and the pyramid theory. During this time they progressed from the *property noticing* to *structuring* levels of understanding, but they were not able to use these generalizations or the pyramid theory to justify they could create *all* of the four-high towers from the three-high towers.

The Doubling Theory began in the *property noticing* when the girls realized they could remove the top block of each of the four-high towers to create the three-high towers (i.e. removing the top block from RRBB would result in RBB). This led them into the *formalizing* level of understanding as they generalized that could “match up” the eight three-high towers with eight of the corresponding four-high towers (Fig. 9, pg. 58) . When prompted, the girls explained why they had placed the towers into that particular arrangement and then placed the remaining four-high towers into that arrangement (Fig. 11 pg. 60). This explanation and extension of the arrangement moved the girls into the *observing* level of understanding as they were now viewing this arrangement as a theory. Immediately after the girls had placed the remaining four-high towers into this particular arrangement they moved into the *structuring* level of understanding when Ashley justified that they had added a red block and a blue block onto each of the three-high towers in order to create four-high towers. It was only *after* the girls had arranged the towers in that particular way that they could justify the relationship between the three-high and four-high towers. This arrangement also helped the girls to justify that the number of towers would double from one height to the next because adding a red block or blue block onto each tower would create two more towers in the next height.

After developing the Doubling Theory, the girls applied the properties of this theory as they justified how many  $n$ -high towers could be built. Although it took the girls a couple of attempts to create the correct expression to represent the number of  $n$ -high towers, the girls based their expressions on the physical properties of the Doubling Theory of adding a red or blue block on top of each tower. They developed the correct expression,  $2^n$ , by focusing on a more abstract property of the Doubling Theory, that the actual number of towers would double from one height to the next. Here, the girls abstracted the properties of their Doubling Theory in order to create an algebraic expression that would account for all of the  $n$ -high towers.

The progression of understanding was similar for the girls as they developed these two theories in the first interview, and the manipulatives were significant in promoting this progression of understanding. Each of the generalizations the girls developed was based on the properties that they noticed about the towers. These generalizations then led the girls to arrange the towers in particular ways. These arrangements helped the girls extend and explain their generalizations. They also created theories based on the properties and patterns they saw in the arrangements of the towers. The girls then justified their theories based on the properties of the arrangements. Once they had developed their theories, they abstracted the properties of those theories in order to create algebraic expressions. The girls developed understanding in this interview as they created and justified and abstracted their theories, and these theories were based on the properties and arrangements of the towers.

**Second Interview.** During the second interview, a similar progression of understanding can be seen when the girls made connections between the towers and Pascal's Triangle in order to develop understanding of the rows and entries of Pascal's Triangle. At the beginning of this interview, the girls arranged the three-high and four-high towers according to the Cases

Theory. They worked within the *property noticing* level of understanding when they created numerical charts (Fig. 16 and 17 pg. 72, 73) based upon the properties they noticed about the number of towers in each particular case. They moved into the *formalizing* level when Ashley recognized the numbers in their charts were somehow related to Pascal's Triangle and she generalized that there was symmetry within each of the rows of the triangle. The girls extended this generalization by making the connection between the symmetry they knew existed in Pascal's Triangle and the numbers in their charts. This led them to develop the Row Theory, that the counts of towers within their charts actually represented the rows in Pascal's Triangle (Fig. 19, pg. 74), and helped them move into the *observing* level of understanding.

After the girls developed the Row Theory, they filled out other rows of the triangle, and we see a shift in their progression of understanding. Instead of progressing from *property noticing* through the *structuring* level, the girls stayed within the *property noticing* and *observing* levels of understanding, with an occasional justification that took them into the *structuring* level. When filling out the other rows of Pascal's Triangle, the girls created charts that helped them account for how many towers would be in each case when the towers of that height were arranged according to the Cases Theory. After creating these charts, they filled in the corresponding rows of Pascal's Triangle. Here, the girls were making connections between their Cases Theory and the Row Theory. The girls also made connections between the rows of the triangle and their theory of  $2^n$  when they explained what the sum of the entries of each row would be. The girls were working in the *property noticing* and *observing* levels of understanding as they were making connections between the properties of different theories, and then moved into the *structuring* level briefly when they justified what the sum of each row would be.

Just as in the first interview, the properties and arrangements of the towers were significant in helping the girls to develop generalizations and theories, but the connections that the girls made between theories also played a significant role in helping the girls develop understanding of the rows of Pascal's Triangle. The girls created numerical charts of the three-high and four-high towers based on the properties they noticed as they arranged those towers by case. These charts helped them make a connection between the rows of Pascal's Triangle and the towers, which led to the Row Theory. Once the girls had developed this theory about Pascal's Triangle, they filled in other rows of the triangle. Finally, they made another connection between the rows of Pascal's Triangle and their theory about  $2^n$  in order to justify the sum of each row.

**Third Interview.** In the third interview, the progression of understanding can be seen when the girls made connections between the towers and the addition rule of Pascal's Triangle in order to justify why the addition rule works. Toward the beginning of the interview, the girls were working in the *property noticing* and *formalizing* levels of understanding when they generalized that the addition rule was somehow related to the property of adding a red or a blue block on top of towers in order to create towers that would represent the next row of the triangle. The girls could not progress any further with this generalization, so I prompted them to consider how the towers might be physically placed on Pascal's Triangle. After this, the girls related their Doubling Theory, of adding a red or a blue block on top of each tower, to this new generalization of physically placing the towers on top of Pascal's Triangle. The girls moved into the *observing* level of understanding when Beth explained the Physical Splitting Theory. She provided a specific example by explaining how the property of the Doubling Theory, adding a red and blue block on top RR and BB, would influence where the newly created three-high towers would be

located in adjacent groups in the third row of Pascal's Triangle. After Beth gave this explanation of how the Physical Splitting Theory worked, the girls continued to work in the *observing* or *structuring* levels of understanding throughout the rest of this interview.

After Beth explained the specific examples of RR and BB, she applied the Physical Splitting Theory to BR and RB and developed the Abstract Splitting Theory by making a connection with another property of their Doubling Theory. Beth recalled that the number of towers would double from one row to the next in Pascal's Triangle. She then moved into the *structuring* level of understanding when she justified the Abstract Splitting Theory by explaining that when they applied their Physical Splitting Theory to a group of towers, "half" of the new towers created would be placed into one adjacent group in the row below, while the other "half" of the new towers would be placed in the other adjacent group. She argued that this halving of the towers was why the addition rule worked.

During the third interview, the properties and arrangements of the towers were still important in helping the girls to develop generalizations and theories, but we also see that the connections the girls made between theories about the towers were also significant in facilitating the girls' understanding. The girls created the Physical Splitting Theory, which was based upon a connection they made between a property of the Doubling Theory (adding a red or a blue block on top of towers) and the Row Theory (that the towers, arranged according to the Cases Theory, represented rows of Pascal's Triangle). The girls could not progress in justifying the addition rule until they developed the Physical Splitting Theory and saw that the towers could be physically placed on top of the rows of Pascal's Triangle. In justifying the Physical Splitting Theory, the girls made a connection about another property of the Doubling Theory (that the

number of towers doubled from one height to the next) in order to develop the Abstract Splitting Theory, which helped them justify the addition rule.

**The Role of Manipulatives.** Analysis of these case study data suggests three main ways in which the manipulatives facilitated mathematical understanding of combinatorics and Pascal's Triangle. First, the properties and arrangements of the manipulatives were significant in helping the girls' progress through the levels of understanding. Hinzman (1997) suggested that the ability to rearrange manipulatives and see them in various patterns could help to stimulate students' senses, as they are able to touch and move them. The girls did benefit from touching and moving the towers in this case study, but more importantly, the actual arrangement of the towers helped the girls to develop theories and justify their reasoning throughout all of the interviews. This can especially be seen when the girls were trying to justify the relationship between the three-high and four-high towers. They could not develop any justifications until they had placed the towers into a particular arrangement (Fig 11, pg. 60). This arrangement not only helped them to justify the relationship, but it led to their development of the Doubling Theory, which was significant throughout the rest of the Towers Task and their development of Pascal's Triangle. Several of the theories the girls developed were based upon their arrangement of the manipulatives in particular ways, and they would not have been able to create or justify those theories had the manipulatives not been arranged accordingly.

Moyer (2001) argued that the physical nature of manipulatives does not carry the meaning of the mathematical ideas, and this was true within this case study. Although the towers and particular arrangements of the towers were significant, the girls had to reflect on the properties of the towers and arrangements in order to form generalizations and theories that helped progress their understanding. The towers and arrangements of towers would not have

facilitated the girls' understanding had the girls not noticed and explained the properties and patterns they saw within the towers and arrangements. Making connections between the properties of towers helped the girls to arrange the towers into particular ways.

A second way the manipulatives facilitated understanding was by providing a foundation for the generalizations and theories the girls developed. The literature did not specifically address the significance manipulatives could have upon students creating mathematical generalizations and theories, but in this case study, all of the generalizations and theories the girls created were based on the manipulatives, and helped the girls to progress into the outer levels of understanding. Also significant, were the connections that the girls made between theories. The girls gave meaning to the rows of Pascal's Triangle and justified the addition rule by making connections between different theories based on the towers. These connections helped the girls to explain and justify their reasoning, which led to meaning making and understanding.

Finally, a third way in which manipulatives facilitated understanding was by helping the girls to reason abstractly and justify abstract expressions and ideas. The literature provides some examples of contexts where students may use manipulatives to help them abstract (Chao et al., 2000; Chappel & Strutchens, 2001; Fuson & Briars, 1990), and there are other examples within in this case study. The girls were used the manipulatives as they developed algebraic notation,  $2^n$ , to describe all of the  $n$ -high towers. They abstracted the properties of the Cases Theory in order to develop the Row Theory and give meaning to what the rows and entries in Pascal's Triangle meant in terms of the towers. The girls also created the Abstract Splitting Theory, which helped them justify why the addition rule in Pascal's Triangle works.

## Making Connections Between Theories

The research on the Pirie-Kieren Theory describes *observing* as an outer level of understanding in which students begin to define their formal ideas as algorithms or theories and *structuring* as an outer level where students justify their theories (S. B. Pirie & T. E. Kieren, 1994; Thom & Pirie, 2006). Although the research describes these different levels of understanding, there is no detail as what kind of activities can help students to remain in these outer levels of understanding. Within this case study, the girls developed and justified theories while working in the *observing* and *structuring* levels of understanding, and they consistently stayed within those levels of understanding by making connections between the theories they had developed.

In the first interview, the students progressed through the levels of understanding several different times. After they had created a theory, such as the Cases Theory, they had to fold back to the *property noticing* and *formalizing* levels to create new generalizations and theories to show how the three-high and four-high towers were related as well as how to express the number of  $n$ -high towers they could build. In this interview, the students developed three significant theories (Cases Theory, Doubling Theory, and  $2^n$  Theory), but they did not consistently stay in the *observing* or *structuring* levels of understanding for very long periods of time. A shift occurs in the second and third interview, where the girls stayed in the higher levels of understanding because they made connections between theories instead of folding back to inner levels of understanding.

During the second interview, after the girls had developed and justified the Row Theory and applied it to the third and fourth row of Pascal's Triangle, they stayed within the *observing* level of understanding throughout the majority of the interview with some movement into the



*structuring* level of understanding. The girls created and justified the fifth row of the triangle by connecting the Row Theory, Cases Theory and Doubling Theory. After this they immediately proved what the sum of each row would be, as well as the very top row by using their  $2^n$  Theory. Within the third interview, the girls spent significant time within the outer levels of understanding. They moved from the *formalizing* to *observing* level of understanding when they created the Physical Splitting Theory by making connections between the Doubling Theory and Row Theory. After they developed and justified the Physical Splitting Theory, they remained in the outer levels of understanding for the rest of the interview. The Abstract Splitting Theory was formed from a connection made between a different property of the Doubling Theory and the Row Theory. The girls then created a physical representation of the fourth row of Pascal's Triangle by applying their Row Theory after which, they each justified why particular four high towers were placed there according to the physical and abstract splitting theories.

The girls stayed within the outer levels of understanding for extended periods of time during the second and third interviews because of the connections they made between different theories. The girls flexibly moved between theories they had developed to extend their understanding of the rows of Pascal's Triangle and the addition rule without having to fold back to inner levels of understanding in order to form new ideas or generalizations. Since the girls had spent significant time developing the Cases, Doubling and  $2^n$  Theories in the first interview, those theories were strong enough to provide a basis to form other theories related to Pascal's Triangle in the second and third interviews.

### **Manipulatives Influence on Abstraction**

Some research has suggested that although manipulatives are useful in prompting students' to reason abstractly, that students should progress from using concrete representations

to abstract representations (Bruner, 1966; Piaget, 1952). Ambrose (2002) pointed out that if manipulatives are overused, they might even hinder some students' abilities to abstract. The analysis in this case study suggests that although students did progress from concrete representations to abstract representations, the students continued to use the manipulatives throughout their abstractions and the manipulatives did not hinder the students' ability to do so. The authors of these particular studies viewed abstraction as the ability to form more general and symbolic representations of mathematical ideas using words and/or notation.

Within the first interview, the abstraction that occurred when the students were developing their algebraic notation  $2^n$  was similar to the type of abstraction described in the literature, but the students still used the manipulatives to help them abstract and form a general expression to represent  $n$ -high towers. The first expression they created, "Poss of  $N = (n - 1) \times 2^n$ " was based entirely on their Doubling Theory of building towers of a particular height. Although the reasoning that brought about this expression was accurate, the girls realized that this expression was inaccurate and were able to fix the mistakes to create the expression, "(Poss of  $n - 1) \times 2$ ," that more accurately represented the Doubling Theory. Beth justified this expression by pointing out specific relationships between the four and five-high towers. The girls then shortened its notation to " $2^n = \# \text{ poss}$ ," by focusing on the fact that the number of towers doubled from height to height. The girls formed these algebraic expressions by abstracting the properties of the Doubling Theory. The girls referenced the towers and arrangements of the towers as they developed the first two expressions "Poss of  $N = (n - 1) \times 2^n$ " and "(Poss of  $n - 1) \times 2$ ," but once they started focusing on the recursive, doubling nature of the towers, they did not have to reference the towers or arrangement in order to develop the final expression  $2^n$ . During the initial stages of abstraction, the girls still relied on the manipulatives in order to create their

algebraic expressions, and even though they used the manipulatives at first, it did not hinder their ability to abstract even farther and develop and justify the expression  $2^n$ .

Within the third interview, the abstraction the girls did was different than the kind of abstraction discussed in the literature, and provides an example of students using manipulatives throughout the entire abstraction process. The girls developed the Abstract Splitting Theory in order to justify the addition rule of Pascal's Triangle and abstracted properties of other theories in doing so. Once the girls had physically arranged the towers on top of Pascal's Triangle, they developed the Physical Splitting Theory which helped them see that placing a red block or blue block on top of each tower would influence where the tower was placed in the subsequent row. The abstraction of this Physical Splitting Theory occurred in the generalized language Beth used when extended this theory into the Abstract Splitting Theory. Beth explained that "half" of the new towers formed, when implementing the Physical Splitting Theory, would be "split" between adjacent groups in the next row. Beth went on to show a specific example of how this halving, or splitting, worked with particular towers and further generalized that it could be applied to all towers and accounted for the addition rule. The girls used the manipulatives throughout this whole development of the Abstract Splitting Theory and in order to justify their conception of the addition rule. Unlike the example from the first interview, the girls never stopped using the towers during the abstraction within the third interview, but they still developed an abstract theory that helped them justify the addition rule.

### **Extension of Pirie-Kieren Theory**

Although research on the Pirie-Kieren model for growth of understanding is fairly extensive (Martin, 2008; Meel, 2003; S. Pirie & T. Kieren, 1994; Thom & Pirie, 2006; Warner, 2008), there is more detail that needs to be added as a result the analysis of these data. First, it

was found that students worked within the *property noticing* level of understanding at the same time as they worked within other levels of understanding. The *property noticing* level is described as the level in which students begin to construct relevant properties about a given topic and are able to verify those properties (S. B. Pirie & Kieren, 1992). Although *property noticing* is an inner level of understanding, and is embedded within the outer levels according to the representation circular representation of the Pirie-Kieren model (Fig. 1, pg. 9), there is a lack of detail about how *property noticing* would occur in the outer levels of understanding.

Within this case study, *property noticing* occurred within the outer levels of understanding as it was found in the generalizations, theories and justifications the girls made throughout all of the interviews. All of the theories the girls developed began in the *property noticing* level of understanding when they first noticed some property about the towers or arrangement of the towers. This led them to develop a generalization based on those particular properties. The *property noticing* that occurred within the *formalizing* level of understanding was more sophisticated as the girls were making connections between different properties, which led them to think more formally about their ideas in order to form generalizations.

After creating generalizations, the girls extended those generalizations into theories based on the properties or patterns they noticed about the generalizations. Oftentimes the explanations of their theories related back to certain properties they had noticed about the towers or arrangements that had led them to create generalizations in the first place. Here, the *property noticing* the girls did while working within the *observing* level was even more sophisticated because they were noticing properties and making connections between formal ideas in order to create their theories. Finally, *property noticing* can also be seen when the girls were working

within the *structuring* level of understanding as the justifications of their theories were based on the properties of those particular theories.

The *property noticing* level of understanding is a very important level within the Pirie-Kieren model of understanding. The properties and connections that students make with a particular idea can lead to them to reason more formally and create generalizations. These generalizations can then be extended into theories as students notice more formal properties about the generalizations and build their theories based on those properties. Once a theory is formed, there are also particular properties of that those theories that can then be justified in order to prove whether or not the theory is accurate. Property noticing is the basis for generalizations, which can be extended into theories, which are then justified. This pattern was seen throughout the course of this particular case study and helps to clarify the influence of *property noticing* within the other levels of understanding in the Pirie-Kieren model.

A second aspect of the Pirie-Kieren model that needs to be clarified relates to the *observing* level of understanding. This level is described as the level wherein students look for patterns and connections in their reasoning in order to organize their ideas as theories (S. B. Pirie & T. E. Kieren, 1994). From the description of this level of understanding, it is unclear whether or not students are only able to form and work with one theory, or whether they may form and work with several theories at a time. Analysis of these case study data shows that students can in fact work with multiple theories at a time within the *observing* level of understanding. During the second and third interviews, there were several times that the girls were making connections between different theories, and proceeded to develop new theories while still working in the *observing* level.

Overall, the analysis of these data clarifies how manipulatives facilitated the growth of one group of high school students' mathematical understanding of combinatorics and Pascal's Triangle. First, the properties and arrangements manipulatives were extremely significant in the progression through the levels of understanding. Second, the students developed generalizations and theories based upon the manipulatives, and third, the manipulatives helped the students to reason abstractly. Important extensions to the Pirie-Kieren Theory were also a result of this analysis.

## Chapter 6: Implications

There are several implications that result from this case study of how manipulatives facilitate the growth of mathematical understanding. First, this study adds to previous theories of how manipulatives are useful in promoting student learning. Second, this study has important implications for teachers who implement manipulatives in their classrooms as well as those who design mathematical manipulatives. Third, this study can direct further study pertaining to manipulatives and the growth of mathematical understanding.

### Implications for Theory

As discussed in prior chapters, prior research has demonstrated how manipulatives are useful for students, but failed to address how manipulatives can facilitate mathematical understanding. This study provided explicit examples that demonstrate how the properties and physical arrangements of manipulatives prompted students to develop mathematical understanding as they traversed through different levels of understanding. The manipulatives also provided a foundation for the students as they formed mathematical generalizations and theories, which also lead to their growth of understanding. This study also provided examples of how students can effectively use manipulatives to help them develop understanding of abstract mathematical concepts, and that the manipulatives supported their ability to abstract.

### Implications for Teachers

In addition to the implications this study has for theory, it also holds important implications for practicing teachers. The context of this case study provided students ample time to develop mathematical generalizations and theories based off of the properties and arrangement of the manipulatives they were working with. Practicing teachers also need to provide their students with enough time to fully explore the properties and various ways the manipulatives

could be arranged in order to form generalizations and theories. Teachers should contemplate the flexibility of particular manipulatives as they are choosing which manipulatives to implement in their classrooms. The manipulatives in this study were flexible enough to be applied across different mathematical concepts, and the students' ability to arrange and then rearrange the manipulatives into patterns helped the students to develop mathematical theories and connect those theories across multiple concepts, which led to the growth of the students' mathematical understanding. This study provided examples where students abstracted while continuing to use manipulatives, so practicing teachers should be cautious when deciding how long to let their students use manipulatives. In some instances, it may be better to allow students to use the manipulatives while they are abstracting, as was the case in this particular case study.

This study also has implications for those experts who design mathematical manipulatives. The design of mathematical manipulatives should take into account the manipulability and flexibility of the manipulatives across different contexts. The flexibility of the manipulatives within this case study allowed the students to arrange them in particular ways, which led to their growth of understanding. The students may not have made as many connections between different concepts and theories had the manipulatives not been as easy to arrange in various ways. Also, because the properties of the particular manipulatives in this study were significant in helping the students to progress, the design of future manipulatives should emphasize the kind of properties the manipulatives hold.

### **Future Study**

This particular case study provides several important implications for the field of mathematics education research and practicing teachers, but also suggests a need for further research regarding manipulatives and the growth of mathematical understanding. First, additional



examples need to be provided that clarify the manipulatives role in facilitating mathematical understanding. The high school students in this case study effectively developed mathematical generalizations and theories based on the properties and arrangements of the manipulatives, which led to the students' growth of understanding. Although the students in this case study were able to use the manipulatives effectively, their use of the manipulatives may have been heavily influenced by the particular task they were engaged with. Although completion of the Towers Task does not necessarily depend on students using manipulatives, the context of the task promotes a clear use of manipulatives if teachers choose to implement them. More research needs to be done to analyze other tasks, where the use of manipulatives is not as clearly defined, in order to determine if the roles of manipulatives in facilitating understanding are similar to the results of this study. Also, additional research should address other types of manipulatives and groups of students that would provide similar influences of the properties and arrangements of manipulatives.

Second, this study provided a counter example to the claim that manipulatives may hinder students ability to abstract, but more research needs to be done in order to support this counter example. It cannot be stated that in all cases, students may be abstract while using manipulatives at the same time, but in this particular study the students abstracted while using the manipulatives simultaneously. More research could be done to determine whether this instance of abstracting and using manipulatives simultaneously is possible in other contexts and with other types of manipulatives.

Third, further research needs to be done in order to clarify other aspects of the Pirie-Kieren Theory for growth of mathematical understanding. This case study clarified how *property noticing* could be identified throughout outer levels of understanding, as well as the fact that

students could develop and use several theories at one time within the *observing* level of understanding, but more research needs to be done to address other details of the theory. One point to consider is how to distinguish between different types of theory building within the *observing* level of understanding, and whether different types of theory building are evidence of more sophisticated activity within the *observing* level. A second point to consider is how some inner levels of understanding, such as *formalizing*, can be seen more explicitly in outer levels such as *observing* and *structuring*. Another aspect that should receive further study is whether or not certain levels of understanding, such as *image making*, *property noticing* and *formalizing*, may be skipped by students if they develop new theories that are formed from connections between other theories they already developed in the *observing* level.

Overall, this case study has addressed the question as to how manipulatives can facilitate the growth of students' mathematical understanding of combinatorics and Pascal's Triangle. It has been shown that the properties and arrangements of towers, led to the development of mathematical generalizations and theories. These theories provided a basis for the students' developing understanding and were aided by the students' ability to make connections between theories and to abstract while using the manipulatives simultaneously. The results of this case study have important implications to the field of mathematics education, both for researchers and practicing teachers, and also show a need for further research pertaining to manipulatives and understanding.

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